

HOMEWORK 6

This problem set is due April 1st... seriously. Feel free to use Sage on the first two.

Reading Read 3.2 by Friday, 3.3 by Monday, 3.4 by Wednesday.

Problem 1 Consider the curve $E_d : y^2 = x^3 + dx$. Exercise 3.7(c) on pg 105 is incorrect. Find the mistake(s) and correct the table. Challenge: can you prove your proposed table is correct?

Problem 2 For each of the following curves, determine the structure of the subgroup of points of finite order. Also, determine the structure of the group and finite generators and relation for the group.

(a) $y^2 = x^3 - 2$

(b) $y^2 = x^3 + 8$

(c) $y^2 = x^3 - 33339627x + 73697852646$

(d) $y^2 + xy - 5y = x^3 - 5x^2$

(e) $y^2 = x^3 - 219x + 1654$

(f) $y^2 + 7xy = x^3 + 16x$

Problem 3 The height function $H(x)$ ($x \in \mathbb{Q}$) can act strangely. For any integer $\kappa \geq 1$ define

$$R(\kappa) = \{q \in \mathbb{Q} : H(q) \leq \kappa\}.$$

Clearly $\#R(\kappa) \leq (\kappa + 1)^2$. But this is overcounting.

(a) Give a bound for $\#R(\kappa)$ of the form

$$\#R(\kappa) \leq \alpha\kappa^2 + \beta\kappa + \gamma$$

where $\alpha \leq 3/2$.

(b) Explain how you would find a better and better bound for $\#R(\kappa)$ (i.e., with α getting smaller and smaller). Describe an algorithm that you could use to find these improvements.

(c) Problem 3.1(b) is the limit of this process. Challenge: solve 3.1(b).