Lecture 8 – Counting Arguments/Inclusion-Exclusion

In lecture 7 we looked into some identities that $\binom{n}{k}$ obeyed. Today we'll make use of them. Recall that $\binom{n}{k}$ is the number of ways of choosing k element subsets from an n element set. Recall the (possibly obvious) multiplication principle: if $E_1, E_2, \ldots, E_k$ are k independent events and if each event E_i has n_i possible outcomes the there are $n_1 n_2 \cdots n_k$ possible outcomes for the collection of k events.

It's common to counts collections of things in two different ways to get identities. See the first example.

Let $A_1, \ldots, A_n$ be a bunch of sets. Let's examine $|A_1 \cup \cdots \cup A_n|$. If we wrote $|A_1 \cup \cdots \cup A_n| = |A_1| + \cdots + |A_n|$ we would have over counted elements in $A_i \cap A_j$. So, we have to subtract off one copy of these elements in these intersections. But then elements in triple intersections are no longer being counted, so we add them back in. And so on.

$$|A_1 \cup \cdots \cup A_n| = \sum |A_i| - \sum |A_i \cap A_j| + \cdots + (-1)^n |A_1 \cap \cdots A_n|.$$

Examples

1. Given p and q positive coprime integers. Prove that

$$\lfloor \frac{p}{q} \rfloor + \lfloor \frac{2p}{q} \rfloor + \cdots + \lfloor \frac{(q-1)p}{q} \rfloor = \lfloor \frac{q}{p} \rfloor + \lfloor \frac{2q}{p} \rfloor + \cdots + \lfloor \frac{(p-1)q}{p} \rfloor.$$

Proof. Think of the integer points inside the rectangle formed by the four points $(0,0)$, $(q,0)$, $(0,p)$ and (p,q). There are $(p-1)(q-1)$ such points. Since p and q are coprime, none of these points lie on the diagonal line from $(0,0)$ to (p,q). Half of them lie above and half below the line.

Now count such points in a different way: the equation of the line is $y = \frac{p}{q}x$ and for each $0 < k < q$ there are $\lfloor kp/q \rfloor$ points with x coordinate k. Therefore there are

$$\lfloor \frac{p}{q} \rfloor + \lfloor \frac{2p}{q} \rfloor + \cdots + \lfloor \frac{(q-1)p}{q} \rfloor$$

points counted this way. So

$$\lfloor \frac{p}{q} \rfloor + \lfloor \frac{2p}{q} \rfloor + \cdots + \lfloor \frac{(q-1)p}{q} \rfloor = (p-1)(q-1)/2.$$

Since the RHS of the equality is unchanged if we switch p and q the desired equality is true.

□

2. An alphabet consists of the letter $a_1, a_2, \ldots a_n$. Prove that the number of all words that contain each of these letters twice but with no consecutive identical letters is equal to

$$\frac{1}{2^n}\left[(2n)! - \binom{n}{1}2(2n-1)! + \binom{n}{2}2^2(2n-2)! - \cdots + (-1)^n 2^n n!\right]$$

Proof. The number of such words without the restriction about consecutive letters is

$$\frac{(2n)!}{(2!)^n}$$

since there are $2n$ distinct letters in these words and each a_i can be switched with the other.

Denote by A_i the set of words formed with the n letters, each occuring twice, for which the two letters a_i appear next to each other. Then we want to count

$$\frac{(2n)!}{2^n} - |A_1 \cup \cdots \cup A_n|.$$

Using the Inclusion-Exclusion principle, we need to count, for example, the number of elements in $|A_{i_1} \cap \cdots \cap A_{i_k}|$ for some indices $i_1, \ldots, i_k$; $k \le n$. Collapse the consecutive letters a_{i_j} $1 \le j \le k$. In effect, then, we are computing the number of words in which $a_{i_1}, \cdots, a_{i_k}$ appear once and all other words appear twice. This number is

$$\frac{(2n-k)!}{2^{n-k}}$$

since there are $2n-k$ letters in all possible orders and $n-k$ pairs of letters for whom the order doesn't matter. Thus

$$\begin{aligned}|A_1 \cup \cdots \cup A_n| &= \sum_k \sum_{i_1,\ldots,i_k} (-1)^{k-1} |A_{i_1} \cap \cdots \cap A_{i_k}| \\ &= \sum_k (-1)^{k-1} \binom{n}{k} \frac{(2n-k)!}{2^{n-k}}\end{aligned}$$

which proves the identity. □

Homework 8 - Counting

Do all the problems: appetizers should be rather straightforward applications of the ideas covered in class; entrees should be less straightforward and desserts should be hard. Good luck and have fun!

Remember, you need to work on these for an hour and you need to show me some evidence that you did. Try small cases. Plug in smaller numbers. Do examples. Look for patterns. Draw pictures. Use lots of paper. Choose effective notation. Look for symmetry. Divide into cases. Work backwards. Argue by contradiction. Consider extreme cases. Modify the problem. Generalize. Dont be afraid of a little algebra.

1. Appetizers

1: How many integers less than 1000 are not divisible by 2, 3 or 5.

2. Entrees

1: Prove the identity

$$\sum_{k=1}^{n} k \binom{n}{k}^2 = n \binom{2n-1}{n-1}.$$

3. Desserts

1: A 150 by 324 by 375 rectangular solid is made by gluing together 1 by 1 by 1 cubes. An internal diagonal of this solid passes through the interior of how many 1 by 1 by 1 cubes?