

LECTURE 7 – BINOMIAL COEFFICIENTS

The binomial coefficients  $\binom{n}{k}$  counts the number of ways one can choose  $k$  objects given  $n$  objects to choose from. It turns out that

$$(x + 1)^n = \binom{n}{0}x^n + \binom{n}{1}x^{n-1} + \cdots + \binom{n}{n-1}x + \binom{n}{n}.$$

Explicitly,

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}.$$

The recurrence relation

$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$$

relates the binomial coefficients to Pascal's triangle.

**Definition:** Let  $a_0, a_1, a_2, \dots$  be a sequence of interest. The formal power series

$$G(x) := \sum_{n=0}^{\infty} a_n x^n$$

is called the (ordinary) generating function for the sequence.

For example, the generating function for the sequence  $\binom{n}{k}$  is the polynomial  $(x + 1)^n$ .

Sometimes this function can be written in closed form. For example, if the sequence satisfies a second order linear recurrence  $a_{n+1} + ua_n + va_{n-1} = 0$ , then

$$G(x) - a_0 - a_1x + ux(G(x) - a_0) + vx^2G(x) = 0.$$

Solving for  $G(x)$  gives

$$G(x) = \frac{a_0 + (ua_0 + a_1)x}{1 + ux + vx^2}.$$

Using partial fractions (Calc II stuff), if  $r_1, r_2$  are roots of  $1 + ux + vx^2$ ,

$$G(x) = \frac{a_0 + (ua_0 + a_1)x}{(1 - r_1x)(1 - r_2x)} = \frac{\alpha}{1 - r_1x} + \frac{\beta}{1 - r_2x}$$

and since  $1/(1 - r_1x)$  is a geometric series with common ratio  $r_1x$  we get

$$G(x) = \sum_{n=0}^{\infty} \alpha(r_1x)^n + \sum_{n=0}^{\infty} \beta(r_2x)^n = \sum_{n=0}^{\infty} (\alpha r_1^n + \beta r_2^n)x^n.$$

So the  $n$  term of the sequence  $a_n$  is  $(\alpha r_1^n + \beta r_2^n)$ .

For example the generating function for the Fibonacci sequence is

$$G(x) = 1/(1 - x - x^2)$$

which leads to the exact formula.

## EXAMPLES

1. Let  $m$  and  $n$  be integers such that  $1 \leq m \leq n$ . Prove that  $m$  divides the number

$$n \sum_{k=0}^{m-1} (-1)^k \binom{n}{k}.$$

*Proof.* Let's try and find an explicit formula for this sum. We apply the recurrence formula from the first page to get

$$\begin{aligned} n \sum_{k=0}^{m-1} (-1)^k \binom{n}{k} &= n \sum_{k=0}^{m-1} (-1)^k \left( \binom{n-1}{k} + \binom{n-1}{k-1} \right) \\ &= n \sum_{k=0}^{m-1} (-1)^k \binom{n-1}{k} - n \sum_{k=0}^{m-2} (-1)^k \binom{n-1}{k} \\ &\quad \text{(n-1 choose -1 doesn't make sense and we change the limits of summation)} \\ &= n(-1)^{m-1} \binom{n-1}{m-1} \quad \text{(most of the summands cancel)} \\ &= m(-1)^{m-1} \binom{n}{m} \quad \text{(multiply by } m/m) \end{aligned}$$

□

2. Prove that

$$\sum_{j=0}^n \binom{n}{j} 2^{n-j} \binom{j}{\lfloor j/2 \rfloor} = \binom{2n+1}{n}.$$

*Proof.* Note that  $\binom{j}{\lfloor j/2 \rfloor}$  is the constant term of  $(1+x)(x+1/x)^j$ . It follows that the sum is equal to the constant term in

$$\begin{aligned} \sum_{j=0}^n \binom{n}{j} 2^{n-j} (1+x)(x+1/x)^j &= (1+x) \sum_{j=0}^n \binom{n}{j} 2^{n-j} (x+1/x)^j \\ &= (1+x)(2+x+1/x)^n \\ &= \frac{1}{x^n} (1+x)(1+2x+x^2)^n \\ &= \frac{1}{x^n} (1+x)^{2n+1} \end{aligned}$$

whose constant term is  $\binom{2n+1}{n}$ .

□

## HOMEWORK 7 - BINOMIAL COEFFICIENTS

Do all the problems: appetizers should be rather straightforward applications of the ideas covered in class; entrees should be less straightforward and desserts should be hard. Good luck and have fun!

Remember, you need to work on these for an hour and you need to show me some evidence that you did. Try small cases. Plug in smaller numbers. Do examples. Look for patterns. Draw pictures. Use lots of paper. Choose effective notation. Look for symmetry. Divide into cases. Work backwards. Argue by contradiction. Consider extreme cases. Modify the problem. Generalize. Don't be afraid of a little algebra.

## 1. APPETIZERS

1: Compute the sums

$$\sum_{k=1}^n k \binom{n}{k} \text{ and } \sum_{k=1}^n \frac{1}{k+1} \binom{n}{k}.$$

## 2. ENTREES

1: Consider the triangular  $n \times n$  matrix

$$A_n = \begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ 0 & 1 & 1 & \cdots & 1 \\ 0 & 0 & 1 & \cdots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix}.$$

Compute  $A^k$  for  $k$  a positive integer.

## 3. DESSERTS

1: If  $k$  and  $m$  are positive integers, prove that the polynomial

$$(x^{k+m} - 1)(x^{k+m-1} - 1) \cdots (x^{k+1} - 1)$$

is divisible by

$$(x^m - 1)(x^{m-1} - 1) \cdots (x - 1)$$

in such a way that the quotient has integer coefficients.