

LECTURE 5 – SEQUENCES

Some useful facts:

Terminology: A **sequence** is an infinite list of numbers $a_0, a_1, \dots, a_n, \dots$.

If you can find an expression for a_n in terms of a_k for $k < n$, you are said to have found a **recursive** description of the sequence. If you can find an expression for a_n only in terms of n , you are said to have found the **explicit** description of the sequence. The term a_n is called the **general term**.

Limit definition: A sequence $\{a_n\}_n$ converges to a finite limit L if and only if for every $\epsilon > 0$ there exists an $n(\epsilon)$ such that for ever $n > n(\epsilon)$, we have $|a_n - L| < \epsilon$.

Squeezing: If $a_n \leq b_n \leq c_n$ for all n and if $\{a_n\}_n$ and $\{c_n\}_n$ converge to the finite limit L , then $\{b_n\}_n$ converges to L as well. Moreover if $\{a_n\}_n$ tends to infinity, $\{b_n\}_n$ does, too.

Weierstrass's Theorem: A monotonic bounded sequence of real numbers is convergent.

Linear Recurrences: Let $a_n = b_1 a_{n-1} + b_2 a_{n-2} + \dots + b_k a_{n-k}$ (think, for example, of the Fibonacci sequence). Let

$$P(x) = x^k - b_1 x^{k-1} - \dots - b_{k-1} x - b_k$$

be the characteristic polynomial of the recurrence and let $\lambda_1, \dots, \lambda_t$ be its roots and let λ_i have multiplicity m_i . Then there are some constants β_{ij} so that

$$x_n = \sum_{i=1}^t \sum_{j=0}^{m_i-1} \alpha_{ij} \binom{n}{j} \lambda_i^{n-j}.$$

Note that the above is an explicit description of the recurrence. For the Fibonacci sequence $F_0 = 0, F_1 = 1$ and $F_{n+1} = F_n + F_{n-1}$ note that $P(x) = x^2 - x - 1$ (its roots are $\lambda_{1,2} = (1 \pm \sqrt{5})/2$). Then the above says that

$$\begin{aligned} 0 = F_0 &= \alpha_{10} \binom{0}{0} \lambda_1^0 + \alpha_{20} \lambda_2^0 \\ 1 = F_1 &= \alpha_{10} \binom{1}{0} \lambda_1^1 + \alpha_{20} \lambda_2^2. \end{aligned}$$

From this we deduce that $\alpha_{10} = -\alpha_{20} = -1/\sqrt{5}$. So we can write

$$F_n = 1/\sqrt{5}(((1 + \sqrt{5})/2)^n - ((1 - \sqrt{5})/2)^n).$$

EXAMPLES

- Find the general term of the sequence given by $a_0 = 3, a_1 = 4$ and

$$(n+1)(n+2)a_n = 4(n+1)(n+3)a_{n-1} - 4(n+2)(n+3)a_{n-2}.$$

Proof. Divide through by the product $(n+1)(n+2)(n+3)$ to get

$$a_n/(n+3) = 4a_{n-1}/(n+2) - 4a_{n-2}/(n+1).$$

Define the new sequence $b_n = a_n/(n+3)$. This sequence satisfies the recursion

$$b_n = 4b_{n-1} - 4b_{n-2}.$$

This characteristic polynomial $x^2 - 4x + 4$ which has a double root at 2. Knowing that $b_0 = 1$ and $b_1 = 1$, we deduce that $b_n = 2^n - n2^{n-1}$. Changing back to a_n we get

$$a_n = (n+3)2^n = n(n+3)2^{n-1}.$$

□

2. Prove that $\lim_{n \rightarrow \infty} \sqrt[n]{n} = 1$.

Proof. As in the definition of the limit define the sequence $\{a_n\} = \{\sqrt[n]{n} - 1\}$. The sequence is positive so I need to find a sequence that lies above it so that I can apply the squeeze theorem. Now

$$n = (1 + a_n)^n = 1 + \binom{n}{1}a_n + \binom{n}{2}a_n^2 + \cdots + \binom{n}{n-1}a_n^{n-1} + a_n^n$$

and thus $n > \binom{n}{2}a_n^2 = \frac{n}{n-1}2a_n^2$. From this we get

$$a_n \leq \sqrt{\frac{2}{n-1}}$$

and by squeezing, we're done.

□

3. Prove that the sequence

$$a_n = \sqrt{1 + \sqrt{2 + \sqrt{3 + \cdots + \sqrt{n}}}}$$

is convergent.

Proof. The sequence is increasing so we need to show that it's bounded from above. Factor out a $\sqrt{2}$ to get

$$a_n = \sqrt{2} \sqrt{\frac{1}{2} + \sqrt{\frac{2}{4} + \sqrt{\frac{3}{8} + \cdots + \sqrt{\frac{n}{2^n}}}}} < \sqrt{2} \sqrt{1 + \sqrt{1 + \sqrt{1 + \cdots + \sqrt{1}}}}$$

where there are n radicals. Now if $b_n = \sqrt{1 + \sqrt{1 + \sqrt{1 + \cdots + \sqrt{1}}}}$, $b_{n+1} = \sqrt{1 + b_n}$.

We see that $b_1 = 1 < 2$ and if $b_n < 2$, then $b_{n+1} < \sqrt{1+2} < 2$. Prove by induction that $b_n < 2$. Therefore, $a_n \leq 2\sqrt{2}$ for all n and by Weierstrass's theorem it is convergent. □

HOMEWORK 5 - SEQUENCES

Do one of each kind of problem: appetizers should be rather straightforward applications of the ideas covered in class; entrees should be less straightforward and desserts should be hard. Good luck and have fun!

Remember, you need to work on these for an hour and you need to show me some evidence that you did. Try small cases. Plug in smaller numbers. Do examples. Look for patterns. Draw pictures. Use lots of paper. Choose effective notation. Look for symmetry. Divide into cases. Work backwards. Argue by contradiction. Consider extreme cases. Modify the problem. Generalize. Don't be afraid of a little algebra.

1. APPETIZERS

1: Write down an explicit description of the sequence

$$1, 2, 2, 3, 3, 3, 4, 4, 4, 4, 5, 5, 5, 5, \dots$$

2: Find a formula for the general term of the sequence defined recursively by $a_1 = 1$, $a_n = a_{n-1} + n$ if n is odd, $a_n = a_{n-1} + n - 1$ if n is even.

2. ENTREES

1: Let $a = 4k - 1$ for some integer k . Prove that for any positive integer n , the number

$$1 - \binom{n}{2}a + \binom{n}{4}a^2 - \binom{n}{6}a^3 + \dots$$

is divisible by 2^{n-1} .

2: Define the sequence $\{a_n\}$ recursively by $a_1 = 1$ and

$$a_{n+1} = \frac{1 + 4a_n + \sqrt{1 + 24a_n}}{16} \text{ for } n \geq 1.$$

Find an explicit formula for a_n .

3. DESSERTS

1: Consider the sequence $\{u_n\}$ defined by $u_0 = u_1 = u_2 = 1$ and

$$\det \begin{pmatrix} u_{n+3} & u_{n+2} \\ u_{n+1} & u_n \end{pmatrix} = n!.$$

Show that u_n is an integer for all n .

2: Prove that the sequence $\{a_n\}_{n \geq 1}$ defined by

$$a_n = 1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n} - \ln(n+1) \text{ for } n \geq 1.$$

is convergent.