Lecture 4 – Inequalities

Some useful facts:

Squares of reals: $x \in \mathbb{R}$, $x^2 = 0$ iff $x = 0$.

Cauchy-Schwarz Inequality: $\left(\sum_{k=1}^n a_k^2\right)\left(\sum_{k=1}^n b_k^2\right) \geq \left(\sum_{k=1}^n a_k b_k\right)^2$. It's an equality if there is some λ so that $a_k = \lambda b_k$ for all k.

Triangle Inequalities: Let $F = \mathbb{R}$ or $F = \mathbb{C}$. Then for any $x, y \in F$,

$$|x| - |y| \leq |x + y| \leq |x| + |y|.$$

They're equalities under certain conditions.

Arithmetic Mean - Geometric Mean Inequality: Let $x_1, x_2, \ldots, x_n$ be nonnegative real numbers. Then

$$\frac{x_1 + \cdots + x_n}{n} \geq \sqrt[n]{x_1 \cdots x_n}.$$

It's an equality if the x_i's are all the same.

Examples

1. Prove that for all real numbers

$$2^x + 3^x - 4^x + 6^x - 9^x \leq 1.$$

Proof. Let $a = 2^x, b = 3^x$. Then we want to show

$$a + b - a^2 + ab - b^2 \leq 1.$$

But this is the same as

$$\begin{aligned} a^2 + b^2 - ab - a - b + 1 &\geq 0 \\ (a-b)^2/2 + a^2/2 + b^2/2 - a - b - 1 &\geq 0 \\ (a-b)^2/2 + (a-1)^2/2 + (b-1)^2/2 &\geq 0 \end{aligned}$$

which is always true. □

2. If $a_1 + \cdots + a_n = n$ prove that $a_1^4 + \cdots + a_n^4 \geq n$.

Proof. By C-S,

$$(a_1 + \cdots + a_n)^2 \leq (1 + 1 + \cdots + 1)(a_1^2 + \cdots + a_n^2).$$

Hence $a_1^2 + \cdots + a_n^2 \geq n$. Repeat the process again and get the desired result. □

3. Show that all the real roots of $P(x) = x^5 - 10x + 35$ are negative.

Proof. Let r be a real positive root. The arithmetic mean $(r^5 + 1 + 1 + 1 + 2^5)/5$ is greater than the geometric mean $\sqrt[5]{r^5 \cdot 1 \cdot 1 \cdot 1 \cdot 2^5}$ so by the AM-GM inequality

$$r^5 + 1 + 1 + 1 + 2^5 - 5 \cdot 2 \cdot r \geq 0$$

The inequality is strict since $1 \neq 2$. But this quantity is just $P(r)$ which should be 0. □

Homework 4 - Inequalities

Do one of each kind of problem: appetizers should be rather straightforward applications of the ideas covered in class; entrees should be less straightforward and desserts should be hard. Good luck and have fun!

Remember, you need to work on these for an hour and you need to show me some evidence that you did. Try small cases. Plug in smaller numbers. Do examples. Look for patterns. Draw pictures. Use lots of paper. Choose effective notation. Look for symmetry. Divide into cases. Work backwards. Argue by contradiction. Consider extreme cases. Modify the problem. Generalize. Dont be afraid of a little algebra.

1. Appetizers

1: If a, b, c are positive numbers prove

$$9a^2b^2c^2 \leq (a^2b + b^2c + c^2a)(ab^2 + bc^2 + ca^2)$$

2: Find

$$\min_{a,b\in\mathbb{R}} \max(a^2 + b, b^2 + a).$$

2. Entrees

1: Find all triples of real numbers that are solutions to the system of equations

$$\frac{4x^2}{4x^2+1} = y$$
$$\frac{4y^2}{4y^2+1} = z$$
$$\frac{4z^2}{4z^2+1} = x$$

2: Find all positive integers $n, k_1, \ldots, k_n$ such that $k_1 + \cdots + k_n = 5n - 4$ and

$$\frac{1}{k_1} + \cdots \frac{1}{k_n} = 1.$$

3. Desserts

1: Let $a_1, \ldots, a_n$ be positive numbers that sum to less than 1. Prove that

$$\frac{a_1 \ldots a_n(1 - (a_1 + \cdots a_n))}{(a_1 + \cdots + a_n)(1 - a_1)(1 - a_2)\cdots(1 - a_n)} \leq \frac{1}{n^{n+1}}$$

2: Let $P(x)$ be a polynomial whose coefficients lie in the interval $[1, 2]$ and let $Q(x), R(x)$ be two constant polynomials so that $P(x) = Q(x)R(x)$ with $Q(x)$ having its leading coefficient equal to 1. Then $|Q(3)| > 1$.