Lecture 3 – Polynomials

For what follows, let $P(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots a_1 x + a_0$.
Some useful facts:

Fundamental Theorem of Arithmetic: $P(x)$ factors uniquely over $\mathbb{C}$ as $P(x) = a_n(x - x_1)(x - x_2)\cdots(x - x_n)$.

Coefficients are symmetric polynomials in the roots: With notation as above

$$\begin{aligned} x_1 + x_2 + \cdots + x_n &= -a_{n-1}/a_n \\ x_1x_2 + x_1x_3 + \cdots x_{n-1}x_n &= a_{n-2}/a_n \\ &\vdots \\ x_1x_2\cdots x_n &= (-1)^n a_0/a_n \end{aligned}$$

Derivatives of polynomials: With notation as above

$$\frac{P'(x)}{P(x)} = \frac{1}{x - x_1} + \frac{1}{x - x_2} + \cdots + \frac{1}{x - x_n}.$$

Lucas's Theorem: The zeroes of the derivative $P'(x)$ of a polynomial $P(x)$ lie in the convex hull of the zeroes of $P(x)$.

Irreducibility: A polynomial $P(x) \in \mathbb{Z}[x]$ is called **irreducible over** $\mathbb{Z}[x]$ if there do not exist polynomials $R(x), Q(x) \in \mathbb{Z}[x]$ (other than ± 1) so that $P(x) = Q(x)R(x)$.

Eisenstein's Criterion: Given a polynomial $P(x) \in \mathbb{Z}[x]$ (with notation as above) and suppose there is a prime number p that does not divide a_n, that divides $a_0, a_1, \ldots, a_{n-1}$ and whose square doesn't divide a_0. Then $P(x)$ is irreducible over $\mathbb{Z}[x]$.

Examples

1. Prove that for every prime number p the polynomial

$$P(x) = x^{p-1} + x^{p-2} + \cdots + x + 1$$

is irreducible over $\mathbb{Z}[x]$.

Proof. Note that $P(x) = (x^p - 1)/(x - 1)$. If $P(x)$ were reducible then so would $P(x + 1)$. But

$$P(x) = \frac{(x+1)^p - 1}{x} = x^{p-1} + \binom{p}{1} x^{p-2} + \cdots + \binom{p}{p-2} x + \binom{p}{p-1}$$

Note that $p | \binom{p}{k}$ for k between 1 and $p - 2$; $p^2 \not| \binom{p}{p-1} = p$; $p \not| a_n = 1$. So, by Eisenstein's criterion we're done. □

2. Determine all polynomials $P(x)$ with real coefficients satsifying $P(x^n) = (P(x))^n$.

Proof. Suppose $P(0) \neq 0$ (we'll do the other case later). Substituting $x = 0$ we get $a_0^n = a_0$. Since a_0 is a nonzero real, $a_0 = 1$ if n is even and ± 1 is n is odd. Taking the derivative of what's in the problem, we get

$$nx^{n-1} P'(x^n) = n(P(x))^{n-1} P'(x).$$

If I plug 0 into the left hand side I get 0. Since $P(0) \neq 0$ the right hand side is 0 is $P'(x) = 0$. Therefore $a_1 = 0$. Differentiating again we get $a_2 = 0$, etc. So $P(x) = \pm 1$.

Now, note if $P(x) = x^k Q(x)$ then

$$P(x^n) = x^{nk} Q(x^n) \text{ and } ((P(x))^n = x^{nk} (Q(x))^n.$$

Therefore $Q(x^n) = (Q(x))^n$.

From the above we conclude that $P(x)$ has to be x^n is n is even and $\pm x^n$ is n is odd. □

3. Solve the system

$$x + y + z = 1 \text{ and } xyz = 1$$

if x, y, and z are complex numbers of absolute value equal to 1.

Proof. Can we show $xy + yz + xz = 1$? If so, then x, y and z would be roots of the polynomial

$$t^3 - t^2 + t - 1 = 0.$$

since the coefficients of a polynomial are symmetric polynomials in the roots. Note $x + y + z = 1$ implies $\bar{x} + \bar{y} + \bar{z} = 1$ and since $x\bar{x} = 1$ (and so on) the equation can be written as $1/x + 1/y + 1/z = 1$. Finding a common denominator and using the fact that $xyz = 1$, we're done. □

Homework 3 - Polynomials

Do one of each kind of problem: appetizers should be rather straightforward applications of the ideas covered in class; entrees should be less straightforward and desserts should be hard. Good luck and have fun!

Remember, you need to work on these for an hour and you need to show me some evidence that you did. Try small cases. Plug in smaller numbers. Do examples. Look for patterns. Draw pictures. Use lots of paper. Choose effective notation. Look for symmetry. Divide into cases. Work backwards. Argue by contradiction. Consider extreme cases. Modify the problem. Generalize. Dont be afraid of a little algebra.

1. Appetizers

1: Prove that the polynomial $P(x) = x^{101} + 101x^{100} + 102$ is irreducible over $\mathbb{Z}(x)$.

2: Verify that $\sqrt[3]{20 + 14\sqrt{2}} + \sqrt[3]{20 - 14\sqrt{2}} = 4$.

3: Prove

$$\frac{P'(x)}{P(x)} = \frac{1}{x - x_1} + \frac{1}{x - x_2} + \cdots + \frac{1}{x - x_n}.$$

2. Entrees

1: Prove that the zeroes of the polynomial $P(x) = x^7 + 7x^4 + 4x + 1$ lie inside the disk of radius 2 centered at the origin.

2: Find all polynomials satisfying the functional equation

$$(x + 1)P(x) = (x - 10)P(x + 1)$$

3: Find all polynomials $P(x)$ with integer coefficients satisfying $P(P'(x)) = P'(P(x))$.

3. Desserts

1: Let p be a prime number. Prove that the polynomial

$$P(x) = x^{p-1} + 2x^{p-2} + \cdots + (p - 1)x + p$$

is irreducible in $\mathbb{Z}[x]$.

2: The polynomial $x^4 - 2x^2 + ax + b$ has four distinct real zeroes. Show that the absolute value of each zero is smaller than $\sqrt{3}$.

3: Prove that there are unique positive integers a and n so that

$$a^{n+1} - (a + 1)^n = 2001.$$