Homework 2 - Number Theory

Do one of each kind of problem: appetizers should be rather straightforward applications of the ideas covered in class; entrees should be less straightforward and desserts should be hard. Good luck and have fun!

Remember, you need to work on these for an hour and you need to show me some evidence that you did. Try small cases. Plug in smaller numbers. Do examples. Look for patterns. Draw pictures. Use lots of paper. Choose effective notation. Look for symmetry. Divide into cases. Work backwards. Argue by contradiction. Consider extreme cases. Modify the problem. Generalize. Dont be afraid of a little algebra.

1. Appetizers

1: Show that if $a \equiv b \pmod{n}$ and $b \equiv c \pmod{n}$ then $a \equiv c \pmod{n}$.

2: An old woman went to marke and a horse stepped on her basket and smashed her eggs. The rider offered to pay for the eggs and asked her how many there were. She did not remember the exact number, but when she had taken them two a time there was one egg left (this also happened when she took them 3,4,5, and 6 at a time). But, when she took them 7 at a time, they came out even. What is the smallest number of eggs she could have had?

3: Prove that

$$\phi(n) = n \prod_{p|n} (1 - 1/p).$$

2. Entrees

1: Prove that $36^{36} + 41^{41}$ is divisible by 77.

2: Find the last three digits of 7^{9999}.

3: Let p be a prime number and a, b, c integers that are divisible by p. Prove that the equation $ax^2 + by^2 \equiv c \pmod{p}$ has integer solutions.

3. Desserts

1: Let a and b be two positive integers so that for any positive integer n, $a^n + n$ divides $b^n + n$. Prove that $a = b$.

2: Solve in the positive integers the equation

$$x^{x+y} = y^{y-x}.$$

3: Prove that if $n \geq 3$ prime numbers form an arithmetic progression, then the common difference in the progression is divisible by any prime $p < n$.