This week I'll give examples of two general approaches to solving geometric problems. I assume you know the standard trig identities.

1. Coordinatize the problem

In a circle are inscribed a trapezoid with one side as a diameter and a triangle with two sides parallel to the sides of the trapezoid. Prove that two have the same area.

Proof. . Assume the circle has radius 1. The four vertices of the trapezoid are $(1, 0), (-1, 0), (a, b), (-a, b)$. The top of the triangle is $(0, 1)$ and we now show that the other two vertices of the triangle are $(\pm b, -a)$. The line that forms the side triangle has slope $b/(a-1)$. Now we find where the line from $(0, 1)$ with slope $b/(a-1)$ intersects the circle by finding the intersection of

$$y = b/(a-1)x + 1 \text{ and } x^2 + y^2 = 1.$$

This is a quadratic with rooms $\pm b$. From here one can easily check the areas are equal. □

2. Think about points as being complex numbers:

Let $A_1 \cdots A_n$ be a regular polygon inscribed in a circle of radius of center O and radius r. On they ray OA_1 chose any point P so that A_1 is between O and A_1. Prove that $\prod |PA_i| = |PO|^n - r^n$.

Proof. WLOG assume that A_1 is on the real axis. Then $A_i = r\zeta^i$ for ζ an nth root of 1. Also, let $P = rx$. Then

$$\prod |PA_i| = \prod |rx - r\zeta_i| = r^n \prod |x - \zeta^i| = r^n(x^n - 1) = (rx)^n - r^n$$

and the equality is proved. □

Homework 10 - Geometry and Trig

Do all the problems: appetizers should be rather straightforward applications of the ideas covered in class; entrees should be less straightforward and desserts should be hard. Good luck and have fun!

Remember, you need to work on these for an hour and you need to show me some evidence that you did. Try small cases. Plug in smaller numbers. Do examples. Look for patterns. Draw pictures. Use lots of paper. Choose effective notation. Look for symmetry. Divide into cases. Work backwards. Argue by contradiction. Consider extreme cases. Modify the problem. Generalize. Dont be afraid of a little algebra.

1. Appetizers

1: Given an acute angled triangle ABC with altitude AD, choose any point M on AD and draw BM until it intersects AC at E. Also, draw CM so that is intersects AB at F. Show that $\angle ADE = \angle ADF$.

2. Entrees

1: Let $ABCDEF$ be a hexagon inscribed in a circle of radius r. Show that if $AB = CD = EF = r$, then the midpoints of BC, DE, FA are the vertices of an equilateral triangle.

3. Desserts

1: In a circle of radius 1 a square is inscribed. A circle is inscribed in the square and then an octagon is inscribed in the inner circle. The procedure is repeated, doubling each time the number of sides of the inscribed polygon. Find the limit of the lengths of the radii of the circles.