Homework 1 - Basic Proof Techniques

Do one of each kind of problem: appetizers should be rather straightforward applications of the ideas covered in class; entrees should be less straightforward and desserts should be hard. Good luck and have fun!

Remember, you need to work on these for an hour and you need to show me some evidence that you did. Try small cases. Plug in smaller numbers. Do examples. Look for patterns. Draw pictures. Use lots of paper. Choose effective notation. Look for symmetry. Divide into cases. Work backwards. Argue by contradiction. Consider extreme cases. Modify the problem. Generalize. Dont be afraid of a little algebra.

1. Appetizers

1: Show that

$$1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}$$

and use this to find a formula for the sum of the first n odd numbers.

2: Show that there are two people in New York city who are both not completely bald but who have exactly the same number of hairs on their head.

3: Show that $\sqrt{2}$ is irrational.

2. Entrees

1: Find all continuous positive functions $f : [0,1] \to \mathbb{R}$ so that

$$\int_0^1 f(x)\ dx = 1$$

$$\int_0^1 f(x)x\ dx = \alpha$$

$$\int_0^1 f(x)x^2\ dx = \alpha^2$$

for a fixed $\alpha \in \mathbb{R}$.

2: Prove that the Fibonacci sequence $\{F_n\}$ satisfies

$$F_{2n+1} = F_{n+1}^2 + F_n^2.$$

3: Given any set of 50 distinct positive integers strictly less than 99, prove that two of them add up to 99.

3. Desserts

1: Show that there does not exist a strictly increasing function $f : \mathbb{N} \to \mathbb{N}$ satisfying $f(2) = 3$ and $f(m \cdot n) = f(m) \cdot f(n)$ for all $m, n \in \mathbb{N}$.

2: Prove that for any positive integer $n > 0$ there exists an n digit number divisible by 2^n containing only the digits 2 and 3.

3: Given 9 points inside the unit square, prove that some 3 of them form a triangle whose area does not exceed $\frac{1}{8}$.