Math 241

16 April 2010
Third Midterm

NAME (Print!): ______________________________

Problem	Points	Score
1	20	
2	20	
3	20	
4	20	
5	20	
Total	100	

Problem 1 (20 points): For the following note that if A is a friend of B then B is a friend of A.

(a) Show that in any group of five people, there are two who have the same number of friends within the group.

(b) Show that in any group of people, there are two who have the same number of friends within the group.

I'll do the general case. There are n people so there are two cases: either someone is friends with nobody (in which case a person can have as few as 0 friends or as many as $n-2$ friends or everyone is friends with someone (in which case a person can have as few as 1 friend or as many as $n-1$ friends). In either case there are n pigeons (the people) and $n-1$ holes (the allowed numbers of friends). This means two people have the same number of friends.

Problem 2 (20 points): In a survey on the chewing gum preferences of baseball players, it was found that

- 22 like fruit.
- 25 like spearmint.
- 39 like grape.
- 9 like spearmint and fruit.
- 17 like fruit and grape.
- 20 like spearmint and grape.
- 6 like all flavors.
- 4 like none.

How many players were surveyed?

$22+25+39-9-17-20+6+4 = 50$

Problem 3 (20 points): Shuffle a standard 52-card deck, and deal out a hand of 13 cards. You get 4 points for each Ace in the hand, 3 points for each King, 2 for each Queen, 1 for each Jack, and nothing for the other cards. Let the random variable X denote the total number of points in your hand.

(a) What's the probability that $X = 0$? Please do not simplify your answer.

(b) What's $E(X)$?

Part (a): there are 36 cards that give no points and there are $\binom{36}{13}$ ways to make a hand out of those cards. So the probability is $\binom{36}{13}/\binom{52}{13}$.

Part (b): let X_i be the points given by the ith card in your hand of 13. Then $X = \sum_{i=1}^{13} X_i$ and so $E(X) = \sum E(X_i)$. Now the expected value of a single card is 10/13: with probability 1/13 your card has value 4, with probability 1/13 it has value 3. similarly for 2 and 1. Otherwise it has value 0. So the expected valueof the whole hand is 10/13*13 = 10.

Problem 4 (20 points): Call a ternary string (a string consisting of 0s, 1s and 2s) lovely if every 0 is immediately followed by a 1 and every 1 is immediately followed by a 2. For instance, the strings 0120120120, 01212, and 222201 are all lovely, but 0120112 is not lovely. Let a_n denote the number of ternary strings of length n that are lovely.

(a) Find a recursion relation that defines a_n. Specify initial conditions.

(b) Prove that $a_n \geq (\sqrt[3]{3})^n$. Hint: $(\sqrt[3]{3})^2 + \sqrt[3]{3} + 1 \geq (\sqrt[3]{3})^3$.

Part (a): Lovely strings can start with either a 0, a 1 or a 2. In the first case, it must start with 012, in the second case 12 and in the third just 2. So the number of lovely strings of length n is the sum of the number of lovely strings of length $n-1$ (those thtat start with a 2), the number of lovely strings of legnth $n-2$ (those that start with a 12) and the number of lovely strings of length $n-3$ (those that start with 012). So $a_n = a_{n-1} + a_{n-2} + a_{n-3}$. Moreover, $a_1 = 3$ (0,1,2), $a_2 = 5$ (01,12,20,21,22) and $a_3 = 9$ (012,120,121,122,201,212,220,221,222).

Part (b): When $n = 1$, $a_1 = 3 \geq \sqrt[3]{3}$. Now let $a_k \geq \sqrt[3]{3}^k$. for all $k < n$. Now

$$\begin{aligned} a_n = a_{n-1} + a_{n-2} + a_{n-3} &\geq \sqrt[3]{3}^{n-1} + \sqrt[3]{3}^{n-2} + \sqrt[3]{3}^{n-3} \\ &= \sqrt[3]{3}^{n-3}((\sqrt[3]{3})^2 + \sqrt[3]{3} + 1) \geq (\sqrt[3]{3})^3 \sqrt[3]{3}^{n-3} = \sqrt[3]{3}^n \end{aligned}$$

Problem 5 (20 points): How many strings of six lowercase letters (possibly all the same) from the English alphabet (which has 26 letters) contain

(a) all the letters n, a, t, h, a, n?
$6!$

(b) the letter a?
$26^6 - 25^6$

(c) the letters a and b?
$26^6 - 25^6 - 25^6 + 24^6$

(d) the letters a and b in consecutive positions with a before b and no repeated letters?
$5 \cdot 24 \cdot 23 \cdot 22 \cdot 21 \cdot 20$