Unit Summary: Vectors and calculus on curves

The topics for this exam are the following (reorganized as I understand them):

(1) vector basics
  (a) vectors in $\mathbb{R}^2$ and $\mathbb{R}^3$
  (b) dot product of two vectors
  (c) cross product of two vectors
(2) calculus on curves
  (a) vector valued functions (i.e., functions that take in some number of real numbers and output a vector); e.g,. lines, vector fields
  (b) calculus on curves (e.g, derivatives, integrals, etc)
  (c) arclength
  (d) acceleration and velocity of a path/curve
  (e) line integrals (both scalar and vector valued)

Vector basics

Let's think a little bit about vectors. A vector is really just a compact way to represent a collection of numbers and they have both an arithmetic and geometric existence. My job is to connect those.

idea	alg	geo
add/sub	component by component	tail to head
points vs vectors	displacement vector	points are heads of vectors
dot product	add up comp by comp mult	$\|\|u\|\|\|\|v\|\|\cos\theta$
cross product	gross determinant calc	magnitude is area of parallelogram, direction is $\perp$ to u and v

Here is a list of things that the dot product is useful for

(1) finding angles between vectors
(2) determining if two vectors are perpendicular
(3) finding projection of one vector onto another

Here is a list of things that the cross product is useful for

(1) finding a directions perpendicular to two vectors
(2) finding the area of a parallelogram spanned by two vectors
(3) determining if two vectors are parallel

Calc on curves

Let's think about how the calculus things are related to things I already know from calc i and calc ii. The main difference is that my functions take possibly more than one number as input and produces more than one number as output. E.g., a vector field and a path.

Derivatives and integrals of paths are done component by component just as in calc i and ii. Arc length in calc ii is

$$\int \sqrt{1+f'(x)^2}\ dx$$

where the integrand is just what I get from Pythagorean theorem:

$$\int \sqrt{1+f'(x)^2}\ dx = \int \sqrt{1+(dy/dx)^2}\ dx = \int \sqrt{dx^2+dy^2}$$

where the last thing is just a length between two points on the function. We get the same thing here

$$\int \sqrt{x'(t)^2+y'(t)^2}\ dt.$$

Acceleration in this setting is a little more interesting. Acceleration is a vector and it can be written in terms of components. One component is the tangential component and is the projection of $a(t)$ onto $T(t) =$

$v(t)/||v(t)||$ and the other is the normal component which is the projection of $a(t)$ onto $N(t) = T'(t)/||T'(t)||$. USe the projection formulas to find these.

The very new stuff is this stuff about line integrals, so I'll be a little more careful here. Let's first understand the notation of scalar line integrals:

$$\int_C f(x, y, z)\ ds = \int_a^b f(c(t))||c'(t)||dt.$$

The LHS says restrict f to C and then add up the length of C weighted by f. Maybe f measures linear density and so the integral gives total mass. The RHS is an algebraic interpretation of the LHS: $f(c(t))$ is literally f restricted to C and $||c'(t)||dt$ is rate times time and so is a little piece of distance. The RHS, importantly, is an integral of something that I know how to integrate.

Let's look at vector line integrals:

$$\int_C F(x, y, z) \cdot d\vec{s} = \int_a^b F(c(t)) \cdot \vec{c}'(t)\ dt.$$

The LHS says restrict F to C and then add up all the work done by F along C. The RHS is, like the above scalar linear integrals, an algebraic interpretation of the LHS.

Formulas I should know

put some stuff in here (e.g, projection)

Problems that I should know how to do

Beyond the obvious computational ones, I should how to do some theoretical stuff like on the quiz about vectors, I should know how to do the ones where I'm shooting a tower.