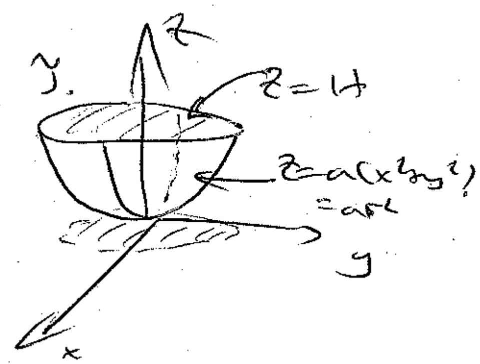
Quiz 7

Problem 1 Let $a > 0$. Find the mass of the solid between the paraboloid $z = a(x^2 + y^2)$ for $0 \leq z \leq H$ where the density of the solid at the point (x, y, z) is given by $\rho(x, y, z) = z$ kg per cu. meter.

Let $\mathcal{Y}$ denote the solid region.
Since density $\times$ volume = mass, we have



$$\text{mass}(\mathcal{Y}) = \iiint_{\mathcal{Y}} \rho(x,y,z)\,dV.$$

$$= \iiint_{\mathcal{Y}} z\,dV$$

we use cylindrical coordinates.

$\mathcal{Y}$: $\cancel{a(x^2+y^2)}\ ar^2 \leq z \leq H$,

$0 \leq r \leq \sqrt{\frac{H}{a}}$

$0 \leq \theta \leq 2\pi$

$$= \int_0^{2\pi}\int_0^{\sqrt{H/a}}\int_{ar^2}^{H} z \cdot r\,dz\,dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^{\sqrt{H/a}} \left[\frac{rz^2}{2}\right]_{z=ar^2}^{z=H} dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^{\sqrt{H/a}} \left(\frac{rH^2}{2} - \frac{a^2r^5}{2}\right) dr\,d\theta.$$

$$= \int_0^{2\pi} \left[\frac{r^2H^2}{4} - \frac{a^2r^6}{12}\right]_{r=0}^{r=\sqrt{H/a}} d\theta$$

$$= \int_0^{2\pi} \frac{H}{a}\cdot\frac{H^2}{4} - \frac{a^2\frac{H^3}{a^3}}{12}\,d\theta$$

$$= \int_0^{2\pi}\left(\frac{H^3}{4a} - \frac{H^3}{12a}\right)d\theta$$

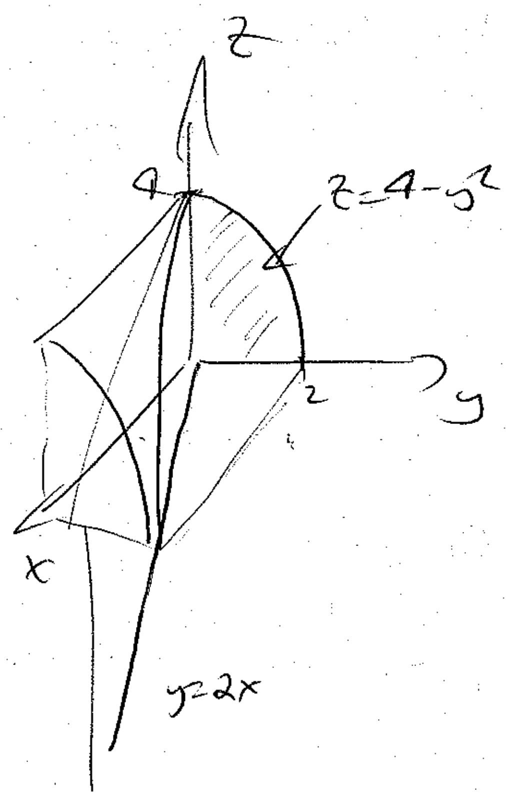
$$= 2\pi\left(\frac{H^3}{4a} - \frac{H^3}{12a}\right) = \frac{H^3\pi}{2a} - \frac{H^3\pi}{6a} = \frac{2H^3\pi}{6a} = \frac{H^3\pi}{3a}$$

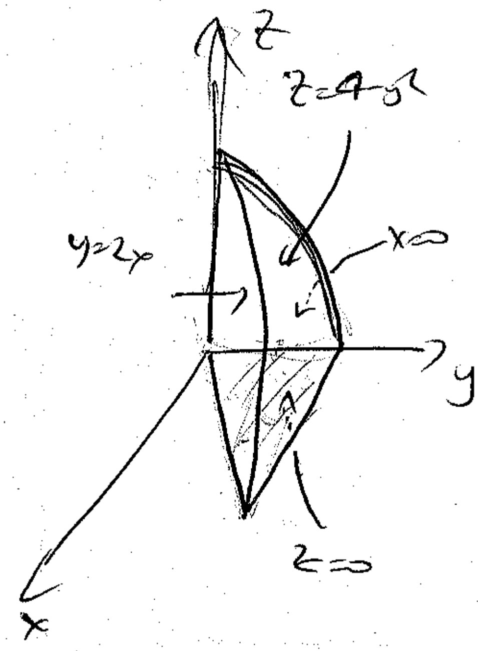
Problem 2 Consider the region W bounded by

$$z = 4 - y^2 \quad y = 2x \quad z = 0 \quad x = 0$$

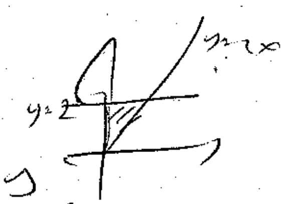
Express the volume of this region as an iterated integral in three different in three different orders:

(1) $dz\,dx\,dy$,
(2) $dx\,dy\,dz$, and
(3) $dy\,dx\,dz$.

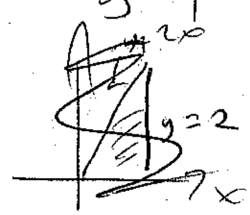




$$\mathrm{vol}(W) = \iiint_W 1\,dV$$



$$= \int_0^2 \int_0^{y/2} \int_0^{4-y^2} dz\,dx\,dy$$



$$= \int_0^4 \int_0^{\sqrt{4-z}} \int_0^{y/2} dx\,dy\,dz$$

$$= \int_0^4 \int_0^{\sqrt{1-\frac{z}{4}}} \int_{2x}^{\sqrt{4-z}} dy\,dx\,dz$$

$z = 4 - 4x^2$

$x^2 = 1 - \frac{z}{4}$