Math 211

April 9, 2012

Sample Third Midterm

NAME: KEY.

Note: No T/F on this one but there will be on the real one.

Problem	Points	Score
1	10	
2	10	
3	10	
4	10	
5	10	
Total	50	

Problem 1 (10 points): Find the average of the distance from the point (x, y) to the point $(0, 0)$ for all points (x, y) in the unit disk $x^2 + y^2 \leq 1$.

$$\underline{avg} = \frac{\iint_D \sqrt{x^2+y^2}\, dA}{\iint_D dA} = \frac{\iint_D \sqrt{x^2+y^2}\, dA}{\pi}$$

$$= \frac{\int_0^1 \int_0^{2\pi} r^2 \, d\theta\, dr}{\cancel{\int_0^1 \int_0^{\pi} dA}\ \pi} = \frac{1}{\pi} \int_0^1 r^2 \cdot 2\pi \, dr$$

$$= \frac{2}{3}$$

Problem 2 (10 points): Consider the ice cream cone region of the top half of the cone $z = \sqrt{x^2+y^2}$ and under the sphere $x^2 + y2 + z^2 = 1$. We want to find its volume.
(1) Set up an integral in rectangular coordinates whose value is that volume.
(2) Also in cylindrical coordinates.
(3) And in spherical coordinates.
(4) Find the volume (points will be deducted if you do a hard one).

(1) $$\iiint_E dV = \iint_D \int_{\sqrt{x^2+y^2}}^{\sqrt{1-x^2-y^2}} dz\, dA$$

$$= \int_{-\frac{1}{\sqrt{2}}}^{\frac{1}{\sqrt{2}}} \int_{\sqrt{\frac{1}{2}-x^2}}^{\sqrt{\frac{1}{2}-x^2}} \int_{\sqrt{x^2+y^2}}^{\sqrt{1-x^2-y^2}} dz\, dy\, dx$$

D: $x^2+y^2+\underbrace{x^2+y^2}_{z^2}=1$, circle of radius $\frac{1}{\sqrt{2}}$

(2) $$\iiint_E dV = \iint_D \int_r^{\sqrt{1-r^2}} r\, dz\, dA$$

$$= \int_0^{\frac{1}{\sqrt{2}}} \int_0^{2\pi} \int_r^{\sqrt{1-r^2}} r\, dz\, d\theta\, dr$$

(3) $$\iiint_E dV = \int_0^{\frac{\pi}{4}} \int_0^{2\pi} \int_0^1 \rho^2 \sin\varphi\, d\rho\, d\theta\, d\varphi$$

(4) $$\iiint_E dV = \int_0^{\frac{\pi}{4}} \sin\varphi\, d\varphi \int_0^{2\pi} d\theta \int_0^1 \rho^2 d\rho$$

$$= \left(1 - \frac{\sqrt{2}}{2}\right)(2\pi)\left(\frac{1}{3}\right)$$

$$= \boxed{\frac{2-\sqrt{2}}{3}\pi}$$

$x = y + 3$

$y = \cancel{3} x \; x - 3$

$1 - y = x$

$y = 1 - x$

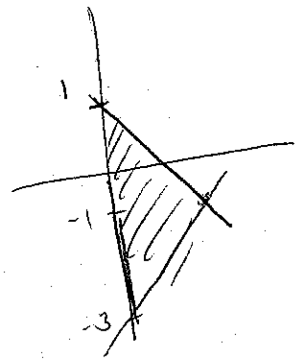
Problem 3 (10 points): Questions about double integrals:

(1) The region, R, for the integral below is an isosceles triangle:

$$\iint_R f(x,y)\,dA = \int_{-3}^{-1}\int_0^{y+3} f(x,y)\,dx\,dy + \int_{-1}^{1}\int_0^{1-y} f(x,y)\,dx\,dy.$$

Sketch R after reversing the order of integration.

$\iint_R f(x,y)\,dA =$



$$\int_0^2 \int_0^{x-3} f(x,y)\,dy\,dx + \int_{2}^{\cancel{0}} \int_0^{1-x} f(x,y)\,dy\,dx = \int_0^2 \int_0^{x-3} f(x,y)\,dy\,dx$$

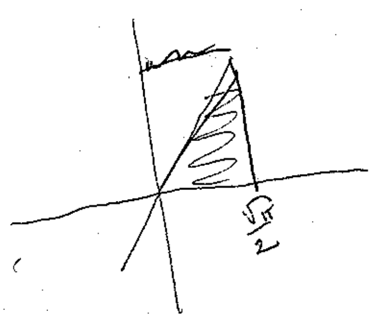
$$+ \int_0^{}\int_{\cancel{1}-x}^{\cancel{0}} f(x,y)\,dy\,dx$$

$$= \int_0^2 \int_{1-x}^{x-3} f(x,y)\,dy\,dx$$

(2) Compute

$$\int_0^{\sqrt{\pi}} \int_{y/2}^{\sqrt{\pi}/2} \sin(x^2)\,dx\,dy.$$

$\frac{y}{2} = x$



$$= \int_0^{\frac{\sqrt{\pi}}{2}} \int_0^{2x} \sin x^2\,dy\,dx$$

$$= \int_0^{\frac{\sqrt{\pi}}{2}} 2x \sin x^2\,dx$$

$$= -\cos(x^2)\Big|_0^{\frac{\sqrt{\pi}}{2}}$$

$$= -\cos\left(\frac{\pi}{4}\right) - (-\cos 0)$$

$$= 1 - \frac{\sqrt{2}}{2}$$

Problem 4 (10 points): Describe the surface parameterized by

$$r(u,v) = \langle u+v, 3u-v, 2u+v \rangle,\ 0 \le u \le 2,\ 0 \le v \le 6$$

and find its area.

$$\iint_S dS = \int_0^2 \int_0^6 \sqrt{42}\, dv\, du$$

$$T_u = \langle 1, 3, 2 \rangle$$

$$T_v = \langle 1, -1, 1 \rangle$$

$$n = T_u \times T_v = \det \begin{pmatrix} i & j & k \\ 1 & 3 & 2 \\ 1 & -1 & 1 \end{pmatrix} = \langle 5, 1, 4 \rangle$$

$$\|T_u \times T_v\| = \sqrt{42}$$

Problem 5 (10 points): Let S be the portion of the paraboloid $z = x^2 + y^2$ above the unit disk with normal pointing up (and into the paraboloid). Consider the vector field $\vec{F}(x, y, z) = \langle 0, 0, z\rangle$ through S and the flux $\iint_S \vec{F} \cdot d\vec{S}$. Explain geometrically why the flux should be positive and verify your answer by carrying out the computation of the flux.

$\iint_S \vec{F} \cdot d\vec{S}$

$\cancel{\Phi} \sum \cdot \langle 0,0,z\rangle \cdot \cancel{n}(a,b,c)$ where $c>0$. So

$\cancel{\Phi} \iint_S \vec{F} \cdot d\vec{S} > 0$.

$S: \langle x, y, x^2+y^2\rangle$

$$\iint_S \vec{F}\cdot d\vec{S} = \iint_D \langle 0,0,x^2+y^2\rangle \cdot \overset{N(x,y)}{|T_x \times T_y|}\, dx\,dy$$

$T_x = \langle 1, 0, 2x\rangle$

$T_y = \langle 0, 1, 2y\rangle$

$$T_x \times T_y = \cancel{\det}\begin{vmatrix} i & j & k \\ 1 & 0 & 2x \\ 0 & 1 & 2y \end{vmatrix}$$

$= i(-2x) \cancel{+} j(2y) + k$

$= \langle -2x, -2y, 1\rangle$

$$= \iint_D \langle 0, 0, x^2+y^2\rangle \langle -2x, -2y, 1\rangle \sqrt{4x^2+4y^2+1}\, dx\,dy$$

$$= \int_0^{2\pi}\int_0^1 r^3\sqrt{4r^2+1}\, dr\, d\theta \qquad u = r^2, \quad du = 2r\,dr$$

$$= \int_0^{2\pi}\int_0^1 \frac{1}{2} u \sqrt{4u+1}\, du\, d\theta \qquad w = 4u+1, \quad dw = 4\,du$$

$$= \int_0^{2\pi}\int_0^1 \frac{1}{2}\,\frac{w-1}{4}\,\cancel{4}\,\cancel{\Phi}\sqrt{w}\,\frac{1}{4}\,dw = \boxed{\frac{1}{120}(1+25\sqrt{5})}$$