Math 211

March 8, 2012

Sample Second Midterm

NAME: KEY

Problem	Points	Score
1	5	
2	9	
3	10	
4	10	
5	6	
6	10	
Total	50	

Problem 1 (5 points): Answer the following True and False questions. You will get 1 point for a correct answer, lose 1 point for a wrong answer and will neither lose nor get a point for leaving it blank. Write out the entire word: TRUE or FALSE

(1) $\lim_{(x,y)\to(0,0)} \frac{x^2-3xy+y^2}{(x-y)^2} = 1$ FALSE

(2) The level curves for the surface $z = y^2 - x^2$ are hyperbolas FALSE

(3) If the tangent plane to the graph of $f(x,y)$ at the point $(x_0, y_0, f(x_0, y_0))$ is horizontal, then $(\nabla f)(x_0, y_0) = \vec{0}$. TRUE

(4) Consider the line $\vec{\ell}(t) = (1+t, 2-t, 3+2t)$. All the following planes
  (a) $x + y = 6$
  (b) $x + y = 5$
  (c) $2y + z = 6$
  (d) $2y + z = 5$
  (e) $2x + 4y + z = 0$
are parallel to $\vec{\ell}(t)$. TRUE

(5) Let $f(x,y) = e^{x^2 \sin(y)}$. Then $\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}$ at the point $(1, \pi/4)$. TRUE

(1) $\lim_{(x,y)\to(0,0)} \frac{x^2-3xy+y^2}{(x-y)^2} = \lim_{(x,y)\to(0,0)} \frac{x^2}{x^2} = 1$
along $y=0$

$\lim_{(x,y)\to(0,0)} \frac{\cancel{\lim}\ x^2-3xy+y^2}{(x-y)^2} = \lim_{(x,y)\to(0,0)} \frac{x^2+3x^2+x^2}{4x^2} = \frac{5}{4}.$
along $y=-x$

(2) $z=0$ level curve are lines

(3) horizontal plane $\Rightarrow$ normal vector $\parallel \hat{k}$ $\Rightarrow$ $\Pi: z = f(x_0, y_0)$
$\Rightarrow f_x, f_y = 0 \Rightarrow \nabla f = \vec{0}$.

(4) normal $\perp$ direction $\Rightarrow$ $\ell \parallel \Pi$:
$\langle 1,1,0\rangle \cdot \langle 1,-1,2\rangle = 0$ a, b
$\cancel{\langle}$ $\langle 0,2,1\rangle \cdot \langle 1,-1,2\rangle = 0$ c, d
$\langle 2,4,1\rangle \cdot \langle 1,-1,2\rangle = 0$. e

(5) mixed partials are equal.

Problem 2 (9 points): Find these derivatives:

(1) Let $f(x,y,z) = x^2 + y^2 + z^2$ and $\vec{c}(s,t) = (s\cos t, e^{st}, s^2 - t^2)$. Find $\frac{\partial f}{\partial s}$.

$$\frac{\partial f}{\partial s} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial s} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial s}$$

$$= (2x)(\cos t) + 2y\, se^{st} + 2z \cdot 2s.$$

sub in $x = s\cos t$, $y = e^{st}$, $z = s^2 - t^2$ if you want.

(2) Consider $f(x,y,z) = 5x^2 + 7y^4 + x^2z^2$. Find the rate of change of f in the direction $\langle 1,1,1\rangle$.

$$D_{\vec{u}} f(P) = \nabla f\big|_P \cdot \vec{u} \qquad \vec{u} = \frac{\langle 1,1,1\rangle}{\sqrt{3}}$$

$$\nabla f = \langle 10x + 2xz^2, 28y^3, 2x^2z\rangle.$$

$$D_{\vec{u}} f(x,y,z) = \frac{1}{\sqrt{3}}\langle 10x + 2xz^2; 28y^3, 2x^2z\rangle.$$

(3) Consider $x^2/4 + y^2/36 + z^2/9 = 1$. Find $\frac{\partial z}{\partial x}$.

Implicit: $\frac{\partial}{\partial x}$ both sides.

$$\frac{x}{2} + \frac{2z}{9}\frac{\partial z}{\partial x} = 0 \Rightarrow \frac{\partial z}{\partial x} = -\frac{x}{2}\cdot\frac{9}{2z} = \frac{-18x}{4z}.$$

Problem 3 (10 points): Questions about curves on surfaces:

(1) The plane $y = 1$ intersects the graph of $z = 3x^2 y - xy^3 + 5y^2$ in a curve. The tangent line to this curve at the point $(1, 1, 7)$ passes through a point $(0, 1, c)$. What is c?

@ $y=1$, this curve is

$$z = 3x^2 - x + 5. \qquad \frac{dz}{dx} = 6x - 1$$ for this curve, and so has tangent line of slope 5.

$$z - 7 = 5(x-1) \quad @\ y=1$$

$$\text{so } z = 5x + 2,\ y = 1.$$

at $x=0$, $y=1$, we see $z = 2$.

(2) A surface S is given by $z = 4 - x^2 - y^2$.

(a) Find the equation of the tangent plane of S at the point $(1, 1, 2)$.

(b) Let C be the curve where S intersects the surface defined by $x^2 y = 1, x > 0$. Think of C as a trail in a terrain described by S. How steep is the trail at the point lying directly above $(x, y) = (1, 1)$? Give the answer as an angle related to the horizontal plane.

(a) $$z - z_0 = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

$$f_x(x,y) = -2x \qquad f_y(x,y) = -2y$$

$$z - 2 = -2(x-1) - 2(y-1) \Rightarrow \boxed{2x + 2y + z = 6.}$$

(b) slope $\tan\theta$ = directional derivative of $f(x,y)$ at $(1,1)$ in the direction given by the curve C. i.e.,

$$\tan\theta = D_{\vec{u}} f(1,1) = \nabla f(1,1) \cdot \vec{u}$$

where $\vec{u}$ is a unit vector that is tangent to C projected to xy plane.

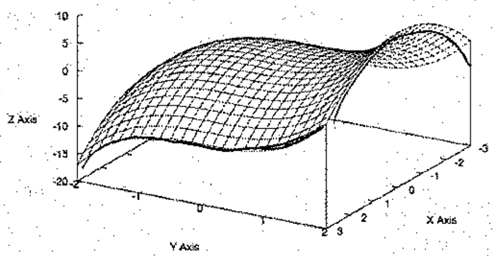
$$y = \frac{1}{x^2} \Rightarrow \vec{r}(t) = \langle t, \tfrac{1}{t^2} \rangle, \quad \vec{r}\,'(t) = \langle 1, -\tfrac{2}{t^3} \rangle,$$ point $(1,1)$ corresponds to $t=1$.

$$\vec{u} = \frac{\vec{r}\,'(1)}{\|\vec{r}\,'(1)\|} = \frac{1}{\sqrt{5}} \langle 1, -2 \rangle.$$

$$\nabla f(x,y) \cdot \vec{u} = \frac{2}{\sqrt{5}} \Rightarrow \tan\theta = \frac{2}{\sqrt{5}} \qquad \theta = \arctan \frac{2}{\sqrt{5}}.$$

Problem 4 (10 points): Some graphing problems:

(1) Match the graph to the correction function:

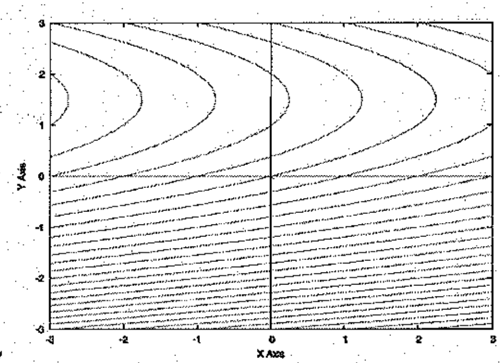


(a) $f(x, y) = y^2$
(b) $f(x, y) = x^2$
(c) $f(x, y) = x - y^2$
(d) $f(x, y) = y - x^2$
(e) $f(x, y) = y^3 - x^2$
(f) $f(x, y) = x^2 - y3$
(g) $f(x, y) = y^4 - x^2$
(h) $f(x, y) = x^4 - y^2$
(i) $f(x, y) = y^4 - x$
(j) $f(x, y) = x^4 - y$

fix a y I get ∩ ~ $-x^2$

fix a x I get ~ y^3

(2) Match the level curves to the correct function



(a) $f(x, y) = x^2$
(b) $f(x, y) = y^2$
(c) $f(x, y) = x^2 - y^2$ ← hyperbolas
(d) $f(x, y) = x^2 + y^2$ ← circles
(e) $f(x, y) = x + y^2$ → $-y^2 = x$ vertex at (0,0)
(f) $f(x, y) = y + x^2$
(g) $f(x, y) = y + x^2 + 3x$
(h) $f(x, y) = y + x^2 - 3x$
(i) $f(x, y) = x + y^2 + 3y$
(j) $f(x, y) = x + y^2 - 3y$

One of these.

$k = x + y^2 + 3y$

$x = -y^2 \mp 3y + k$

$= -(y^2 + 3y - k)$

$= -(y + \frac{3}{2})^2$

centers shifted to the left

Problem 5 (6 points): Suppose that the magnitude of two vectors are measured as 5 and 6 respectively and the maximum error in the measurement of each is 0.4. Suppose that the angle between the two of them is measured at $\pi/3$ radians with the maximum error in that measurement being 0.02 radians. Use linear approximation to find the maximum error in computing the dot product of the two vectors using the given data.

$P(\cancel{v_m}\, l_1, l_2, \theta) = l_1 l_2 \cos\theta$

$\cancel{d\theta =}\; dP = P_{l_1}\, dl_1 + P_{l_2}\, dl_2 + P_\theta\, d\theta$

$= l_2 \cos\theta\, dl_1 + l_1 \cos\theta\, dl_2 - l_1 l_2 \sin\theta\, d\theta$

$= 6\cos\left(\frac{\pi}{3}\right)(0.4) + 5\cos\left(\frac{\pi}{3}\right)(0.4) - 5\cdot 6\cdot \sin\frac{\pi}{3}(0.02)$

$= \boxed{2.2 - 0.3\sqrt{3}}$

Problem 6 (10 points): Let $g(x,y) = \sqrt[3]{xy}$.

(1) Is g continuous at $(0,0)$?

Yes: Can plug in $(0,0)$

(2) Calculate g_x and g_y when $xy \neq 0$.

$$\frac{\partial g}{\partial x} = \sqrt[3]{y} \cdot \frac{1}{3} x^{-2/3} \qquad \frac{\partial g}{\partial y} = \sqrt[3]{x}\ \frac{1}{3} y^{-\frac{2}{3}}$$

(3) Show that $g_x(0,0)$ ad $g_y(0,0)$ exist by assigning values to them.

$$\cancel{\frac{\partial g}{\partial x} = \lim_{h \to 0} \frac{g(x+h,y) - g(x,y)}{h} = 0.}$$

$\frac{\partial g}{\partial x} = \frac{\partial g}{\partial y} = 0$ since intersecting g w/ the planes $x=0, y=0$, gives the 0 fcn.

Possibly too hard

(4) Are g_x and g_y continuous at $(0,0)$?

No: $y = 0 \Rightarrow \lim_{x \to 0} g_x = 0$

$c(t)$ $(t^{3/2}, \sin t)$ along this path $\frac{\partial g}{\partial x} = \lim_{t \to 0} \frac{1}{3} \sin t\, t^{-1} = \frac{1}{3}$.

(5) Does the graph of $z = g(x,y)$ have a tangent plane at $(0,0)$?

No: along the line $y=x$, $y = x^{2/3}$ which has a corner.

So there's no tangent plane.

(6) Is g differentiable at $(0,0)$?

No, b/c there's no tangent plane