Math 211

April 6, 2012

Third Midterm (part 1)

NAME: KEY

Problem	Points	Score
1	11	
2	9	
3	5	
Total	25	

Problem 1 (11 points): Prove some geometric formulas:

(1) Prove, using an integral, that the volume of a sphere of radius R is given by $\frac{4}{3}\pi R^3$.

Let W be the sphere of radius R.

$$\iiint_W dV = \int_0^{2\pi}\int_0^{\pi}\int_0^{R} \rho^2 \sin\varphi \, d\rho\, d\varphi\, d\theta$$

$$= \int_0^{2\pi}\int_0^{\pi} \frac{R^3}{3} \sin\varphi \, d\varphi\, d\theta$$

5

$$= \frac{R^3}{3}\int_0^{2\pi} -\cos\varphi \Big|_0^{\pi} d\theta$$

$$= \frac{2R^3}{3}\int_0^{2\pi} d\theta = \frac{4\pi R^3}{3}$$

(2) Prove, using an integral, that the surface area of a sphere of radius R is given by $4\pi R^2$.

$$\iint_S dS = \int_0^{2\pi}\int_0^{\pi} R^2 \sin v \, dv\, du = 2R^2\int_0^{2\pi} du$$

$$= \boxed{4\pi R^2}$$

6

$G(u,v) = R\langle \cos u \sin v, \sin u \sin v, \cos v\rangle$ for $0 \le u \le 2\pi$, $0 \le v \le \pi$

$T_u = R\langle -\sin u \sin v, \cos u \sin v, 0\rangle$

$T_v = R\langle \cos u \cos v, \sin u \cos v, -\sin v\rangle$

$$T_u \times T_v = R^2 \begin{pmatrix} i & j & k \\ -\sin u \sin v & \cos u \sin v & 0 \\ \cos u \cos v & \sin u \cos v & -\sin v \end{pmatrix} = R^2\left(i(-\cos u \sin^2 v) - j(\sin u \sin^2 v) + k\,\cancel{(\ldots)} + k(-\sin^2 u \sin v\cos v - \cos^2 u \sin v \cos v)\right)$$

$$\cancel{= R^2\left[i(-\sin^2 u \sin^2 v - \cos^2 u \cos^2 v) + j(-\sin u \sin^2 v)\right) + k}$$

$$= \cancel{-\sin v} - R^2 \sin v \langle \cos u \sin v, \sin u \sin v, \cos v\rangle$$

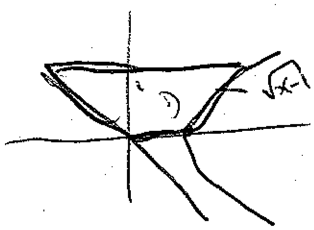
$$\|T_u \times T_v\| = R^2 \sin v \sqrt{\cos^2 u \sin^2 v + \sin^2 u \sin^2 v + \cos^2 v}$$

$$= R^2 \sin v \sqrt{\sin^2(v) + \cos^2 v}$$

$$= R^2 \sin v$$

Problem 2 (9 points): Set up but do not solve the following integrals. Be sure to include some explanation of your thought process.

(1) The volume under the plane $z = y$ and above the region in the xy-plane bounded by the curve $x = y^2 + 1$ and the lines $y = -x$, $y = 0$ and $y = 1$.



$$\iint_D \int_0^y dz\,dA = \int_0^1 \int_{-y}^{y^2+1} \int_0^y dz\,dx\,dy$$

(2) The volume of the region under the half-sphere $z = \sqrt{2 - x^2 - y^2}$ and over the paraboloid $z = x^2 + y^2$. Set it up in the coordinate system most suitable for computations.

$$\iint_D \int_{x^2+y^2}^{\sqrt{2-x^2-y^2}} dz\,dA = \int_0^{2\pi}\int_0^1 \int_{r^2}^{\sqrt{2-r^2}} r\,dz\,dr\,d\theta.$$

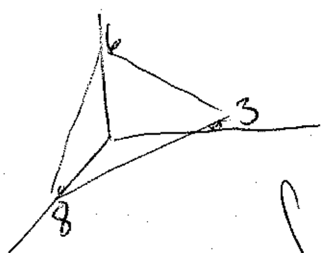
D: intersection of surfaces

$\cancel{x^2+y^2 = \sqrt{2-x^2-y^2}}$ $\quad z^2 = 2 - z$

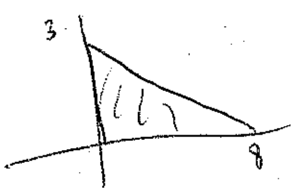
$z^2 + z - 2 = 0 \Rightarrow (z+2)(z-1) = 0 \Rightarrow z = 1$ Since $z > 0$.

@ $z = 1$, $x^2 + y^2 = 1$.

(3) The volume of the tetrahedron bounded by the coordinate planes and the plane $3x + 8y + 4z = 24$.



$$\iint_D \int_0^{\frac{24-3x-8y}{4}} dV = \int_0^8 \int_0^{-\frac{3}{8}x+3} \int_0^{\frac{24-3x-8y}{4}} dz\,dy\,dx$$



$y = -\frac{3}{8}(x-8)$

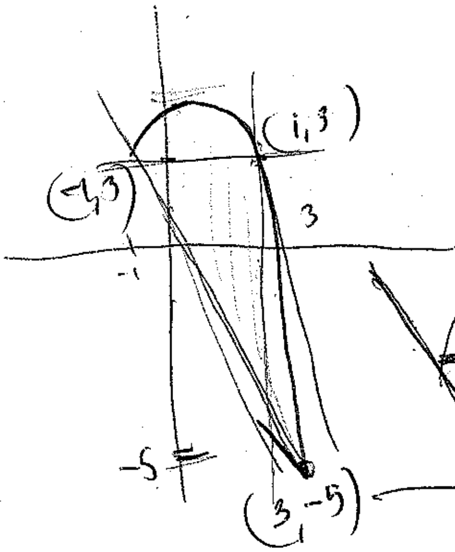
$y = -\frac{3}{8}x + 3$

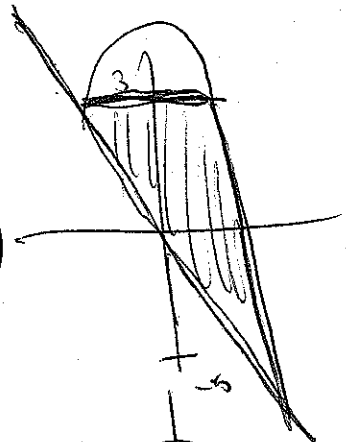
$x = \sqrt{4-y}$

$y = 4-x^2$

Problem 3 (5 points): Change the order of integration for

$$\int_{-5}^{3}\int_{(1-y)/2}^{\sqrt{4-y}} f(x,y)\,dx\,dy + \cancel{\int_{-5}^{3}\int_{-\sqrt{4-y}}^{\sqrt{4-y}} f(x,y)\,dx\,dy}$$





$\frac{1-y}{2} = x$

$1-y=2x$

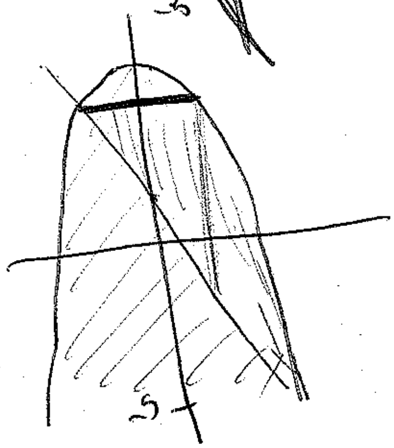
$-2x+1=y$

$-2x+1=4-x^2$

$x^2-2x-3=0$

$(x-3)(x+1)=0$

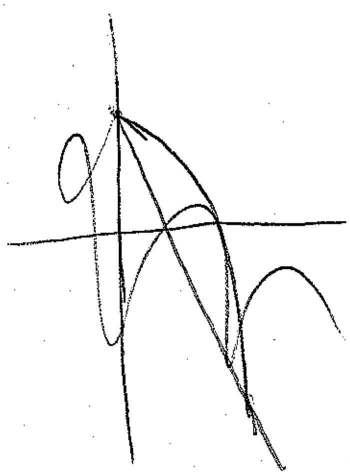
$\int_{-1}^{1}$

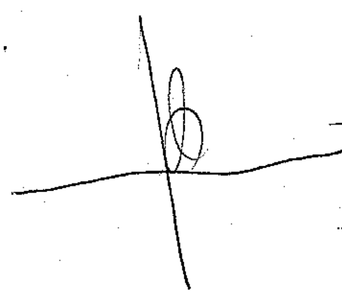


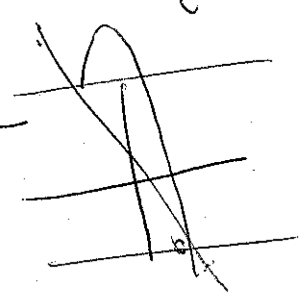
$\int_{-1}^{1}\int_{1-2x}^{3} f(x,y)\,dy\,dx$

$+\int_{1}^{3}\int_{1-2x}^{4-x^2} f(x,y)\,dy\,dx$

$-5 \le y \le 5$







$$\int_{-1}^{1}\int_{1-2x}^{3} + \int_{1}^{3}\int_{1-2x}^{4-x^2}$$