Dot product

We've seen that we can add and subtract vectors, what could vector multiplication possibly mean? Today we'll talk about one kind of vector multiplication: it takes two vectors and gives a scalar in return.

Definition. *Let* $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$ *and* $\mathbf{w} = \langle w_1, w_2, w_3 \rangle$. *Then*

$$\cdot \mathbf{w} = v_1 w_1 + v_2 w_2 + v_3 w_3$$

is the dot product of the vectors $\mathbf{v}$ *and* $\mathbf{w}$.

Here are some simple properties enjoyed by the dot product:

Theorem. *Properties of the dot product:*

(1) $\mathbf{0} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{0} = 0$;
(2) $\mathbf{v} \cdot \mathbf{w} = \mathbf{w} \cdot \mathbf{v}$;
(3) for every scalar λ, $(\lambda \mathbf{v}) \cdot \mathbf{w} = \lambda(\mathbf{v} \cdot \mathbf{w}) = \mathbf{v} \cdot (\lambda \mathbf{w})$;
(4) $\mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) = \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{w}$; *and*
(5) $\mathbf{v} \cdot \mathbf{v} = ||\mathbf{v}||^2$.

Problem 1: Prove any one of the 5 properties above.

This one is done by components. Say, (5):
Let $\vec{v} = \langle v_1, v_2, v_3 \rangle$, then $\vec{v} \cdot \vec{v} = v_1^2 + v_2^2 + v_3^2$ by the definition of the dot product. Notice $\sqrt{v_1^2 + v_2^2 + v_3^2} = \|v\|$, by the definition of length and the identity follows.

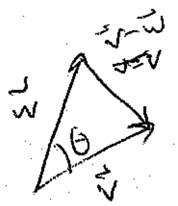
This one is not done by components

Problem 2: Another useful property of the dot product is its geometric interpretation:

$$\mathbf{v} \cdot \mathbf{w} = ||\mathbf{v}|| \, ||\mathbf{w}|| \cos\theta$$

where $0 \leq \theta < \pi$ is the angle between $\mathbf{v}$ and $\mathbf{w}$. Prove this identity in the following steps:

(1) Draw a triangle from $\mathbf{v}$, $\mathbf{w}$ and $\mathbf{v} - \mathbf{w}$ and use the law of cosines (it's in the front of your book) to write down an expression for $||\mathbf{v} - \mathbf{w}||^2$.

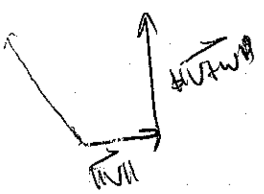


Law of cosines implies.
$\|\vec{v} - \vec{w}\|^2 = \|\vec{v}\|^2 + \|\vec{w}\|^2 - 2\|v\|\|w\| \cos\theta$

(2) Using the properties of the dot product on the first page, write down a different expression for $\|\mathbf{v}-\mathbf{w}\|^2$.

$$\|v-w\|^2 = (v-w)\cdot(v-w) = v\cdot v - 2\cdot v\cdot w + w\cdot w$$
$$= \|v\|^2 - 2\cdot v\cdot w + \|w\|^2$$

(3) Compare the two expressions you wrote down to deduce $\mathbf{v}\cdot\mathbf{w} = \|\mathbf{v}\|\,\|\mathbf{w}\|\cos\theta$. Also, what's true about the dot product of two vectors that are orthogonal (i.e., perpendicular)?



Now

$$\|v\|^2 - 2\cdot v\cdot w + \|w\|^2 = \|v\|^2 + \|w\|^2 - 2\,\|v\|\,\|w\|\cos\theta$$

$$\Rightarrow \quad \vec{v}\cdot\vec{w} = \|v\|\,\|w\|\cos\theta.$$

Problem 3: Let $\mathbf{v}$ and $\mathbf{w}$ be two vectors. Suppose $\|\mathbf{w}\| = 5$, that the sum of $\mathbf{v}$ and $\mathbf{w}$ is perpendicular to $\mathbf{v}$ and has magnitude twice that of $\mathbf{v}$. Find the magnitude of $\mathbf{v}$.

what do I know:

$$\|\vec{w}\| = 5 \quad \Rightarrow \quad \vec{w}\cdot\vec{w} = 25 \ (*)$$

$$(\vec{v}+\vec{w}) \perp \vec{v} \quad \Rightarrow \quad \underset{(**)}{(\vec{v}+\vec{w})\cdot\vec{v} = 0} \quad \Rightarrow \quad \vec{v}\cdot\vec{v} + \vec{w}\cdot\vec{v} = 0$$
$$\Rightarrow \quad \vec{w}\cdot\vec{v} = -\vec{v}\cdot\vec{v} = -\|v\|^2 \ (***)$$

$$\|\vec{v}+\vec{w}\| = 2\|\vec{v}\| \Rightarrow \|\vec{v}+\vec{w}\|^2 = 4\|v\|^2$$
$$\Rightarrow (v+w)\cdot(v+w) = 4\|v\|^2$$
$$\Rightarrow \underbrace{\vec{v}\cdot(\vec{v}+\vec{w})}_{\substack{0 \\ (**)}} + \vec{w}\cdot(\vec{v}+\vec{w}) = 4\|v\|^2$$

$$\vec{w}\cdot\vec{v} + \vec{w}\cdot\vec{w} = 4\|v\|^2$$
$$\overset{(***)\text{ and }(*)}{\Rightarrow} -\|v\|^2 + \|w\|^2 = 4\|v\|^2$$
$$\Rightarrow 25 = 5\|v\|^2$$
$$\Rightarrow \|v\| = \sqrt{5}.$$