Dot product

We've seen that we can add and subtract vectors, what could vector multiplication possibly mean? Today we'll talk about one kind of vector multiplication: it takes two vectors and gives a scalar in return.

Definition. *Let* $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$ *and* $\mathbf{w} = \langle w_1, w_2, w_3 \rangle$*. Then*

$$\mathbf{v} \cdot \mathbf{w} = v_1 w_1 + v_2 w_2 + v_3 w_3$$

is the dot product of the vectors $\mathbf{v}$ *and* $\mathbf{w}$*.*

Here are some simple properties enjoyed by the dot product:

Theorem. *Properties of the dot product:*

(1) $\mathbf{0} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{0} = 0$*;*
(2) $\mathbf{v} \cdot \mathbf{w} = \mathbf{w} \cdot \mathbf{v}$*;*
(3) for every scalar λ*,* $(\lambda \mathbf{v}) \cdot \mathbf{w} = \lambda(\mathbf{v} \cdot \mathbf{w}) = \mathbf{v} \cdot (\lambda \mathbf{w})$*;*
(4) $\mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) = \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{w}$*; and*
(5) $\mathbf{v} \cdot \mathbf{v} = ||\mathbf{v}||^2$*.*

Problem 1: Prove any one of the 5 properties above.

Problem 2: Another useful property of the dot product is its geometric interpretation:

$$\mathbf{v} \cdot \mathbf{w} = ||\mathbf{v}|| \, ||\mathbf{w}|| \cos \theta$$

where $0 \leq \theta < \pi$ is the angle between $\mathbf{v}$ and $\mathbf{w}$.

Prove this identity in the following steps:

(1) Draw a triangle from $\mathbf{v}$, $\mathbf{w}$ and $\mathbf{v} - \mathbf{w}$ and use the law of cosines (it's in the front of your book) to write down an expression for $||\mathbf{v} - \mathbf{w}||^2$.

(2) Using the properties of the dot product on the first page, write down a different expression for $||\mathbf{v} - \mathbf{w}||^2$.

(3) Compare the two expressions you wrote down to deduce $\mathbf{v} \cdot \mathbf{w} = ||\mathbf{v}|| \, ||\mathbf{w}|| \cos\theta$. Also, what's true about the dot product of two vectors that are orthogonal (i.e., perpendicular)?

Problem 3: Let $\mathbf{v}$ and $\mathbf{w}$ be two vectors. Suppose $||\mathbf{w}|| = 5$, that the sum of $\mathbf{v}$ and $\mathbf{w}$ is perpendicular to $\mathbf{v}$ and has magnitude twice that of $\mathbf{v}$. Find the magnitude of $\mathbf{v}$ using the properties about vectors you've proved today.