Conservative vector fields

There are a lot of theorems that allow us to conclude that a given vector field is conservative. Many of the theorems, though, have assumptions. We'll look at this now.

Recall $\vec{F}$ is conservative by definition if there exists a potential function V so that $\vec{F} = \nabla V$.

There's also a theorem that says if $\vec{F}$ is defined on an open connected region, $\vec{F}$ is conservative if and only if it's path independent. This is our first example of an assumption. Take a second to think about why this assumption is reasonable in this context (suppose the region wasn't connected and imagine integrating from one part of the region to the other...).

Problem 1: Prove the following statement: Suppose $\vec{F} = \nabla V = \langle F_1, F_2, F_3 \rangle$ is conservative. Then

$$\frac{\partial F_1}{\partial y} = \frac{\partial F_2}{\partial x}, \ \frac{\partial F_2}{\partial z} = \frac{\partial F_3}{\partial y}, \ \frac{\partial F_3}{\partial x} = \frac{\partial F_1}{\partial z}.$$

$$\frac{\partial F_1}{\partial y} = \frac{\partial}{\partial y}\frac{\partial V}{\partial x}, \quad \frac{\partial F_2}{\partial x} = \frac{\partial}{\partial x}\frac{\partial V}{\partial y}$$

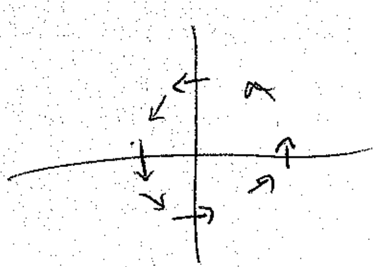
$$\frac{\partial F_2}{\partial z} = \frac{\partial}{\partial z}\frac{\partial V}{\partial y}, \quad \frac{\partial F_3}{\partial y} = \frac{\partial}{\partial y}\frac{\partial V}{\partial z}$$

$$\frac{\partial F_3}{\partial x} = \frac{\partial}{\partial x}\frac{\partial V}{\partial z}, \quad \frac{\partial F_1}{\partial z} = \frac{\partial}{\partial z}\frac{\partial V}{\partial x}.$$

Mixed partials being equal imply that these are equal. ~~Similar~~ Clairaut's Thm

Problem 2: Does the other way work? Consider $\vec{F} = \langle -y/(x^2+y^2), x/(x^2+y^2) \rangle$.

(1) Sketch $\vec{F}$. All unit vectors.



(2) Does $\frac{\partial F_1}{\partial y} = \frac{\partial F_2}{\partial x}$?

$$\frac{\partial}{\partial y}\left(\frac{-y}{(x^2+y^2)}\right) = \frac{(x^2+y^2)(-1) - (-y)(2y)}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2}$$

$$\frac{\partial}{\partial x}\left(\frac{x}{x^2+y^2}\right) = \frac{x^2+y^2 - 2x(x)}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2}.$$

So, Yes.

(3) Is $\vec{F}$ conservative? Consider the unit circle with counterclockwise orientation. Can you argue geometrically that $\vec{F}$ is or isn't conservative?

Angle of tangent vector & vector field vector is 0, so $\vec{F} \cdot d\vec{s} > 0$, so

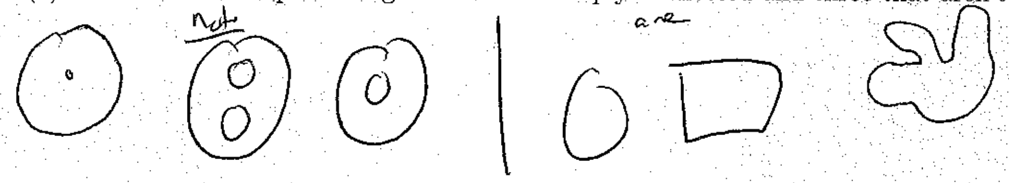
$\int \vec{F} \cdot d\vec{s} > 0$

Problem 3: We say a region D is simply connected if you can carry out the following process without ever leaving D. Draw any a loop anywhere inside of D. Now shrink the loop (without tearing it) down to a point. If your loop never leaves D, we say D is simply connected.

(1) E.g., the region on which the $\vec{F}$ of the previous problem is defined is not simply connected. Why?

hole at $(0,0)$ can't shrink any loop that contains $(0,0)$ to a point.

(2) Draw three examples of regions that are simply connected and three that aren't.



Problem 4: Here's the actual theorem. Let $\vec{F}$ be a vector field defined on a simply connected domain D. If $\vec{F}$ satisfies the cross-partials condition on the previous page, then $\vec{F}$ is conservative. To summarize, why doesn't problem 2 contradict this theorem? B/c the F isn't defined on a simply connected region

Problem 5: Recall $\nabla = \langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \rangle$. Prove if $\vec{F}$ is conservative, then $\nabla \times \vec{F} = \vec{0}$.

$$\nabla \times \vec{F} = \det \begin{pmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{pmatrix} = i\left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right) - j\left(\frac{\partial F_3}{\partial x} - \frac{\partial F_1}{\partial z}\right) + k\left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)$$

$= \vec{0}$ by problem 1.

Problem 6: What about the other way? This theorem is harder to prove and so we won't try: Let $\vec{F}$ be a vector defined on all of $\mathbb{R}^3$ whose components have continuous partial derivatives. If $\nabla \times \vec{F} = \vec{0}$, then F is conservative. Choose a vector field you know is conservative, satisfies the hypotheses of the theorem and use the theorem to show that it's conservative.

$f = x + y + z$

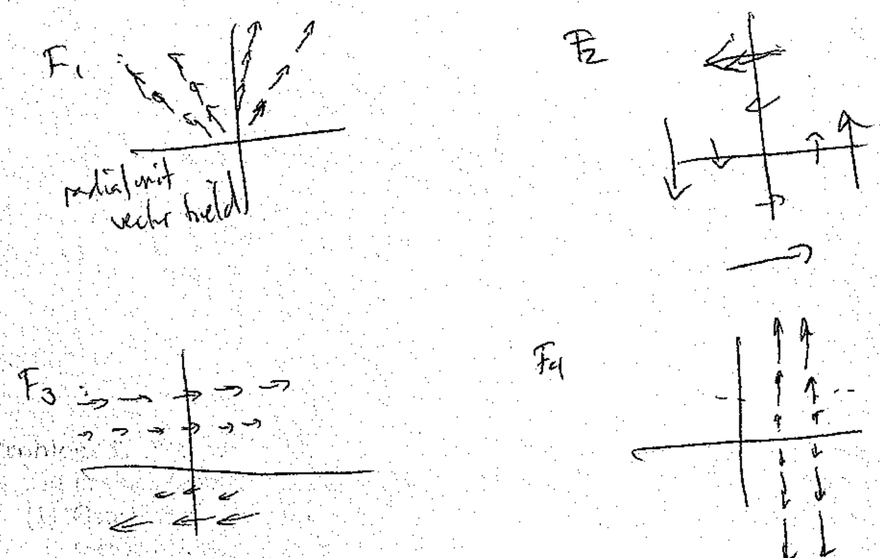
$\nabla f = \langle 1, 1, 1 \rangle$

Notice $\nabla \times F = \det \begin{pmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 1 & 1 & 1 \end{pmatrix} = \vec{0}$.

Problem 7: Now we look at conservative vector fields geometrically. Roughly speaking, if a vector field has not tendency to "swirl around", the vector field should be conservative.

(1) Consider $\vec{F}_1 = \langle x/(x^2+y^2), y/(x^2+y^2)\rangle$, $\vec{F}_2 = \langle -y, x\rangle$, $\vec{F}_3 = \langle y, 0\rangle$, $\vec{F}_4 = \langle \cancel{0, x}\rangle$. Sketch these vector fields.

$\langle 0, y\rangle$



(2) Two of them are conservative and two of them aren't. Justify this statement geometrically and see if it agrees with my comment above about "swirling around".

F_3 & F_2 have some swirling around

F_4 & F_1 don't.

(3) Now show it algebraically.

~~$\vec{F}_4 = \langle 0, y\rangle$~~

$\vec{F}_4 : \dfrac{\partial 0}{\partial y} = \dfrac{\partial}{dx} y$

$0 = 0.$

$\vec{F}_4$ is defined on all of $\mathbb{R}^2$ which is simply connected

So yes, conservative

~~$\vec{F}_1 : \frac{\partial}{\partial y}\frac{x}{x^2+y^2} = \frac{-2yx}{(x^2+y^2)^2} = \frac{\partial}{\partial x}\frac{y}{x^2+y^2}$~~

$\vec{F}_1 : \quad F_1 = \nabla \frac{1}{2}\ln|x^2+y^2|$ can't do it the other way since D for F_1 isn't simply connected

$F_2 \quad \dfrac{\partial(-y)}{\partial y} \neq \dfrac{\partial(x)}{\partial x}$ so F_2 not conservative

$F_3 : \quad \dfrac{\partial}{\partial y}(y) \neq \dfrac{\partial(0)}{\partial x}$ so F_3 not conservative