Some useful facts about modular arithmetic:

Wilson's Theorem: For every prime p, $(p-1)! \equiv -1 \pmod{p}$.

Proof. Consider the numbers $1, 2, 3, \ldots, p-1 \pmod{p}$; and pair numbers a, b up so that $ab \equiv 1 \pmod{p}$. When can $a = b$ in such a pair? If $a^2 \equiv 1 \pmod{p}$ then either $a-1$ or $a+1$ are 0 mod p. I.e., either $a = 1, p-1$. Otherwise a number can be matched with its inverse. I.e., the product $2 \cdot 3 \cdot \cdots \cdot p-2$ is 1 mod p. So $(p-1)!$ is $p-1$. □

Euler ϕ-function: Let $\phi(n)$ be the number of positive integers $1 \leq k \leq n$ so that k and n are relatively prime. Here are some useful properties for this function:

(1) $\phi(n) = n \prod_{p|n}(1 - 1/p)$
(2) if $\gcd(m, n) = 1$, $\phi(mn) = \phi(n)\phi(m)$ (i.e, ϕ is multiplicative),
(3) if p is prime, $\phi(p) = p - 1$,
(4) of $\gcd(a, n) = 1$, $a^{\phi(n)} \equiv 1 \pmod{n}$ – this is Euler' s theorem, the special case when $n = p$, a prime, is Fermat's Little Theorem: $a^{p-1} \equiv 1 \pmod{p}$,
(5) $\phi(n)$ is even for $n > 2$.

Proof. If $n = 2^k$ $(k > 1)$, then $\phi(n) = 2^k(1 - 1/2)$ which is even since $k > 1$. If p is odd, then $\phi(p^k) = p^k(1 - 1/p) = p^k - p^{k-1}$ which is also even. □

Many proofs about the ϕ function proceed in this way: start with a power of two, a power of an odd prime and use multiplicativity.