Some useful facts about sequences:

Terminology: A **sequence** is an infinite list of numbers $a_0, a_1, \ldots, a_n, \ldots$. If you can find an expression for a_n in terms of a_k for $k < n$, you are said to have found a **recursive** description of the sequence. If you can find an expression for a_n only in terms of n, you are said to have found the **explicit** description of the sequence. The term a_n is called the **general term.**

Limit definition: A sequence $\{a_n\}_n$ converges to a finite limit L if and only if for every $\epsilon > 0$ there exists an $n(\epsilon)$ such that for ever $n > n(\epsilon)$, we have $|a_n - L| < \epsilon$.

Squeezing: If $a_n \leq b_n \leq c_n$for all n and if $\{a_n\}_n$ and $\{c_n\}_n$ converge to the finite limit L, then $\{b_n\}$ converges to L as well. Moreover if $\{a_n\}_n$ tends to infinity, $\{b_n\}_n$ does, too.

Weierstrass's Theorem: A monotonic bounded sequence of real numbers is convergent.

Special kinds of Sequences: Here are some sequences that can be defined recursively but can also be written down explicitly:

Geometric: Let $a_n = ra_{n-1}$. Then $a_n = a_1 r^{n-1}$.

Arithmetic: Let $a_n = a_{n-1} + k$. Then $a_n = a_1 + (n-1)k$.

Linear Recurrences: Let $a_n = b_1 a_{n-1} + b_2 a_{n-2} + \cdots + b_k a_{n-k}$ (think, for example, of the Fibonacci sequence). Let

$$P(x) = x^k - b_1 x^{k-1} - \cdots - b_{k-1}x - b_k$$

be the characteristic polynomial of the recurrence and let $\lambda_1, \ldots, \lambda_t$ be its roots and let λ_i have multiplicity m_i. Then there are some constants α_{ij} so that

$$x_n = \sum_{i=1}^{t} \sum_{j=0}^{m_i - 1} \alpha_{ij} \binom{n}{j} \lambda_i^{n-j}.$$

Note that the above is an explicit description of the recurrence. For the Fibonacci sequence $F_0 = 0, F_1 = 1$ and $F_{n+1} = F_n + F_{n-1}$ note that $P(x) = x^2 - x - 1$ (its roots are $\lambda_{1,2} = (1 \pm \sqrt{5})/2$). Then the above says that

$$0 = F_0 = \alpha_{10} \binom{0}{0} \lambda_1^0 + \alpha_{20} \lambda_2^0$$

$$1 = F_1 = \alpha_{10} \binom{1}{0} \lambda_1^1 + \alpha_{20} \lambda_2^2.$$

From this we deduce that $\alpha_{10} = -\alpha_{20} = -1/\sqrt{5}$. So we can write

$$F_n = 1/\sqrt{5}(((1+\sqrt{5})/2)^n - ((1-\sqrt{5})/2)^n).$$

Some useful facts about series

Terminology: A **series** is an infinite sum of the form $a_0 + a_1 + a_2 + \ldots$ and is written $\sum_{i=0}^{\infty} a_i$.

Definition: Let $S_N = a_0 + a_1 + \cdots + a_N$. Then $\sum_{i=0}^{\infty} a_i = \lim_{N \to \infty} S_N$.

Special kinds of Series: Here are some series whose behavior we can describe

Geometric Series: If $|r| < 1$, $\sum_{i=0}^{\infty} ar^i = \frac{a}{1-r}$.

Telescoping Series: Consider $\sum \frac{1}{i(i+1)} = \sum \frac{1}{i} - \frac{1}{i+1}$. Then $S_N = 1 - \frac{1}{i+1}$ and $\lim_{N \to \infty} S_N = 1$.

p-series: If $p > 1$, then $\sum \frac{1}{n^p}$ converges. Otherwise, it diverges.

Alternating series: If $\lim a_n = 0$, then $\sum (-1)^n a_n$ converges.

Divergence test: If $\lim_{n \to \infty} a_n \neq 0$, the series $\sum a_n$ diverges. If the limit is 0, we can't conclude anything.,

Comparison test: If $\sum a_n$ converges and $0 \leq b_n \leq a_n$, then $\sum b_n$ converges. If $\sum b_n$ diverges and $0 \leq b_n \leq a_n/$, then $\sum a_n$ diverges.

Absolute convergence: If $\sum |a_n|$ converges then $\sum a_n$ converges.

EXAMPLES

1. Prove that every positive integer can be written as the sum of distinct Fibonacci numbers.

Induct on n. $1 = F_0 + F_1$.

Let F_k be the ~~small~~ largest fibonacci number that doesn't exceed n and let the hypothesis hold for all integers less than n.

Now $n - F_k = \sum F_{i_j} \Rightarrow n = F_k + \sum F_{i_j}$. We're done unless one of the F_{i_j} is F_k. But F_k is maximal:

$n < F_{k+1} = F_k + F_{k-1} < 2F_k \Rightarrow n - F_k < F_k$

Since $n - F_k < F_k$, then each $F_{i_j} \leq n - F_k < F_k$. □

2. Let k be a positive constant. Suppose the sequence x_i of positive reals has sum k. What are the possible values for the sum of x_i^2?

$\sum x_i = k$: $\left(\sum x_i\right)^2 = \sum x_i^2 + \sum_{i<j} 2x_i x_j$

$k^2 - \sum 2x_i x_j = \sum(x_i^2)$ so $0 < \sum x_i^2 < k^2$.

Now we show every element of $(0, k^2)$ can be reached.

Let $x_0, x_1, \ldots$ be geometric with $\frac{x_1}{x_0} = d$. Then $\sum x_j^2 = \frac{x_0^2}{1-d^2}$

$= \frac{x_0}{1+d} \cdot \sum x_j$ $\qquad \frac{(1)x_0}{1-d} = \sum x_j$

$= \frac{1-d}{1+d} \left(\sum x_j\right)^2$ $\qquad x_0 = (1-d)\sum x_j$

↑

as d increases from 0 to 1

as d ...