

Complex Numbers

We will now take a brief diversion to discuss a very important extension of our idea of “number”. To motivate our discussion we begin by trying to solve a couple of quadratic equations.

Equation A: Find the solutions of $z^2 - z - 1 = 0$. As usual, you start by trying to factor, but you quickly find that you can't. So you resort to the *quadratic formula* to get

$$z = \frac{1 \pm \sqrt{5}}{2}.$$

Equation B: Find the solution of $z^2 - z + 1 = 0$. Here we get

$$z = \frac{1 \pm \sqrt{-3}}{2},$$

and we would conclude that our equation has no solutions because you can't take the square root of a negative number.

But what if it *did* make sense to take the square root of a negative number? Specifically, denote the quantity $\sqrt{-1}$ by the symbol “ i ”. Now, in the solution of **Equation B**, we have

$$z = \frac{1 \pm \sqrt{-1}\sqrt{3}}{2} = \frac{1 \pm i\sqrt{3}}{2} = \frac{1}{2} \pm \frac{\sqrt{3}}{2}i. \quad (1)$$

If you play around with a few more examples, you will quickly realize that if *any quadratic equation* with real coefficients has two solutions that are not both real, then they have the form $x \pm iy$, where x and y are real numbers.

Definitions and terminology. A COMPLEX NUMBER is an expression of the form

$$z = x + iy,$$

where x and y are real numbers. We say that x is the REAL PART of z , denoted $\text{Re}(z)$, and y is the IMAGINARY PART of z , denote $\text{Im}(z)$. (Note that $\text{Im}(z)$ is a *real* number.) Complex numbers such that $\text{Re}(z) = 0$ (such as i and $-i\sqrt{3}$) are called PURELY IMAGINARY NUMBERS.

If $z = x + iy$ is a complex number, then $\bar{z} = x - iy$ is the COMPLEX CONJUGATE of z . Notice that the solutions of the second quadratic equation in (1) are complex conjugates. As suggested earlier, this is true in general: complex solutions (having non-zero imaginary part) to quadratic equations with real coefficients always come in conjugate pairs.

Roots of polynomials. It is a fact that all of the solutions of *any* polynomial equation of *any degree* are complex numbers. Let's have a look what this would mean for the easy example $z^4 = -1$. First, taking square roots, we get $z^2 = \pm\sqrt{-1} = \pm i$. Taking square roots again, we get $z = \pm\sqrt{\pm i}$; we get one solution for each combination of + and - (4 solutions in total). So, if \sqrt{i} is a complex number, it must mean that there exist real numbers x and y such that $\sqrt{i} = x + iy$. How can we find these x and y ? We'll find out in a short while.

Arithmetic. Now that we have extended our notion of “number” to include complex numbers, we need rules for addition, subtraction, multiplication and division of complex numbers. Let $z = x + iy$ and $z' = x' + iy'$ be two complex numbers.

Addition and subtraction are obvious. We define

$$\begin{aligned} z \pm z' &= (x + iy) \pm (x' + iy') \\ &= (x \pm x') + i(y \pm y'). \end{aligned} \tag{2}$$

That is, you just add/subtract the real and imaginary parts separately.

Multiplication is a little trickier. Here we must expand and use the property $i^2 = -1$:

$$\begin{aligned} zz' &= (x + iy)(x' + iy') \\ &= xx' + ix'y' + iyx' + i^2yy' \\ &= (xx' - yy') + i(xy' + yx'). \end{aligned} \tag{3}$$

To invert a complex number, we play a little trick with complex conjugates (this should remind you of the “rationalising the numerator” trick in algebra). Note that

$$z\bar{z} = (x + iy)(x - iy) = x^2 + y^2, \tag{4}$$

a real number! Hence, to invert a complex number z , we multiply top and bottom by its complex conjugate:

$$\frac{1}{z} = \frac{\bar{z}}{z\bar{z}} = \frac{x - yi}{x^2 + y^2} = \frac{x}{x^2 + y^2} - i\frac{y}{x^2 + y^2}. \tag{5}$$

It is easy to divide complex numbers now that we know how to invert them:

$$\frac{z}{z'} = \frac{x + iy}{x' + iy'} = \frac{x + iy}{x' + iy'} \cdot \frac{x' - iy'}{x' - iy'} = \frac{xx' + yy'}{x'^2 + y'^2} + i\frac{x'y - xy'}{x'^2 + y'^2}. \tag{6}$$

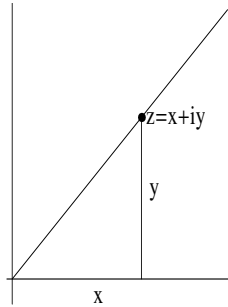
WARNING: It's much easier to just *do* division and multiplication when you have to. By that, we mean remember how it works, but don't try to remember the formula.

Exercises.

1. Express each of the following in the form $x + iy$:
 - (a) $(2 + 3i)(5 + 7i)$

- (b) $(1 + i)^2 + (1 + i)$
 (c) $\frac{3 - 4i}{1 + 2i}$
 (d) $\left(\frac{2 + i}{6i - (1 - 2i)}\right)^2$.
 (e) $((4 - i)^2 - 4)i$.
- Calculate i^n for $n = -1, -2, -3, -4$. What pattern do you observe? What is the value of i^{-36} ? Of i^{-41} ?
 - If z_1 and z_2 are complex numbers, show that $\overline{z_1 z_2} = \overline{z_1} \cdot \overline{z_2}$.
 - If the roots of the equation $z^2 + 2bz + c = 0$ are the complex numbers $p \pm iq$, find expressions for p and q in terms of b and c .
 - Find all solutions to $z^2 + 2z + 5 = 0$.
 - Show that $\operatorname{Re}(iz) = -\operatorname{Im}(z)$.
 - Simplify the complex numbers $i^{4k}, i^{4k+1}, i^{4k+2}$ and i^{4k+3} , where k is any integer.

Visualising complex numbers. When we think of a real number, we think of it as lying on the “real line”. Analogously, we think of complex numbers as lying on the “complex plane”. Since real numbers are a subset of complex numbers, we should be able to see the real numbers in the complex plane. The same should be true of the purely imaginary numbers. Using rectangular coordinates, we view the real numbers as the x -axis, and the purely imaginary numbers as the y -axis. Now the complex number $z = x + iy$ is “plotted” on the plane as the point (x, y) .



Recall from equation (4) that if $z = x + iy$, then $z\bar{z} = x^2 + y^2$. Observe that this is the square of the distance of z from the origin in the complex plane. We define this distance to be the MODULUS of z , and denote it $|z|$. Thus

$$|z| = \sqrt{z\bar{z}} = \sqrt{x^2 + y^2}. \quad (7)$$

Exercise.

- Show that $|zz'| = |z||z'|$.

We will soon see that it can be very useful to represent complex numbers as points in the

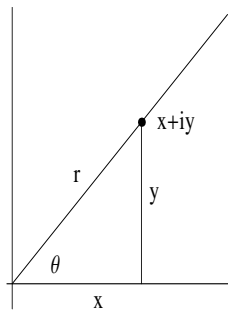
plane in “polar form”. Before proceeding let’s refresh our memory about polar coordinates.

Exercises.

9. The following points are given in rectangular coordinates; find their polar coordinates: $(1, 0)$; $(0, 2)$; $(-1, 1)$; $(0.2, -0.2)$; and $(-3, 1)$.
 10. The following points are given in polar coordinates; find their rectangular coordinates: $(1, 0)$; $(2, \pi)$; $(\sqrt{2}, 5\pi/4)$; $(3, \pi/2)$; and $(1, 1)$.
 11. Sketch the graphs of the functions $r = 2$ and $\theta = 3\pi/4$, given in polar coordinates.
 12. For each of the following sets of inequalities, sketch the region of the plane containing all points whose polar coordinates satisfy the inequalities.
 1. $2 \leq r \leq 3$.
 2. $r \leq 2$ and $0 \leq \theta \leq \pi/2$.
 3. $5 \leq r \leq 10$ and $\pi/3 \leq \theta \leq \pi/2$.
-

Euler’s formula. We have just seen that the complex number $z = x + iy$ corresponds to the point $P = (x, y)$ using rectangular coordinates. Hence, if $P = (r, \theta)$ in polar coordinates, we have

$$z = x + iy = r \cos \theta + ir \sin \theta = r(\cos \theta + i \sin \theta). \tag{8}$$



Exercise.

13. If $z = r[\cos \theta + i \sin \theta]$ and $z' = r'[\cos \theta' + i \sin \theta']$, show that

$$zz' = rr'[\cos(\theta + \theta') + i \sin(\theta + \theta')],$$

and derive a similar formula for $\frac{z}{z'}$.

It turns out that it is often easier to calculate with the polar form of a complex number due to a striking relationship between exponential functions, trigonometric functions and complex numbers.

Euler's Formula: $e^{i\theta} = \cos \theta + i \sin \theta$

Thus, from equation (8), any complex number can be written in the form

$$z = re^{i\theta}, \quad \text{for } r \geq 0 \text{ and } 0 \leq \theta < 2\pi \quad (9)$$

\sqrt{i} as a complex number. Earlier we pondered how to write \sqrt{i} in the form $x + iy$. In rectangular coordinates i corresponds to the point $(0, 1)$. In polar coordinates, it corresponds to the point $(1, \pi/2)$; that is $i = \cos(\pi/2) + i \sin(\pi/2)$. Using Euler's Formula, we can write $i = \cos(\pi/2) + i \sin(\pi/2) = e^{i\pi/2}$. We can now take the square root:

$$\begin{aligned} \sqrt{i} = \sqrt{e^{i\pi/2}} &= (e^{i\pi/2})^{1/2} \\ &= e^{i\pi/4} && \text{(property of exponential functions)} \\ &= \cos(\pi/4) + i \sin(\pi/4) && \text{(Euler's Formula again)} \\ &= \frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}. \end{aligned} \quad (10)$$

You should verify this by squaring the complex number on the right.

Exercise.

14. Find and plot all 3rd roots of 8.

First proof of Euler's formula¹: Consider a complex number z lying on the unit circle $r = 1$ of the complex plane. It has polar form $z = \cos \theta + i \sin \theta$; that is, you can regard the unit circle as a function of θ . Taking the derivative with respect to θ we get

$$\frac{dz}{d\theta} = -\sin \theta + i \cos \theta = i \cos \theta + i^2 \sin \theta = i(\cos \theta + i \sin \theta) = iz.$$

Solving this differential equation using separation of variables, we get

$$\int \frac{dz}{z} = \int i d\theta.$$

Hence $\ln z = i\theta + C$, so that $z = Re^{i\theta}$, where $R = e^C$. When $\theta = 0$, $z = \cos(0) + i \sin(0) = 1 = Re^{i \cdot 0} = R$, so that $R = 1$, as required. ♠

Second proof of Euler's formula²: We can prove Euler's theorem in completely

¹This proof uses – without justification – some basic facts about the calculus of complex numbers.

²This proof assumes – again without justification – that functions of complex numbers have power series expansions.

different way using Taylor series. Recall the series

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots$$

Substituting $x = i\theta$ into the series for e^x , we get

$$\begin{aligned} e^{i\theta} &= 1 + i\theta + \frac{(i\theta)^2}{2} + \frac{(i\theta)^3}{3!} + \frac{(i\theta)^4}{4!} + \dots \\ &= 1 + i\theta - \frac{\theta^2}{2} - i\frac{\theta^3}{3!} + \frac{\theta^4}{4!} + \dots \end{aligned}$$

On the other hand

$$\begin{aligned} \cos \theta + i \sin \theta &= \left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \dots\right) + i\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \dots\right) \\ &= 1 + i\theta - \frac{\theta^2}{2} - i\frac{\theta^3}{3!} + \frac{\theta^4}{4!} + \dots \end{aligned}$$

Euler's Formula now follows. ♠

We record one final, very useful formula concerning the polar form of a complex number. For any positive integer n , we have the following.

De Moivre's Theorem: $[r(\cos \theta + i \sin \theta)]^n = r^n (\cos n\theta + i \sin n\theta)$

Exercises:

15. Express each of the following complex numbers in polar form $z = re^{i\theta}$: $1 + i$; $-3 - 4i$; and $-1 + 3i$.
16. Express each of the following complex numbers in rectangular form $z = x + iy$: $\sqrt{-i}$; $(1 + i)^{100}$; $(-4 + 4i)^{2/3}$; and $(\sqrt{5} + 2i)^{\sqrt{2}}$.
17. Use Euler's Formula to prove the trigonometric identity $\cos 2\theta = \cos^2 \theta - \sin^2 \theta$.
18. Use Euler's Formula to show that $\cos x = \frac{e^{ix} + e^{-ix}}{2}$ and $\sin x = \frac{e^{ix} - e^{-ix}}{2i}$.
19. Prove De Moivre's Theorem.
20. Write the complex numbers $\frac{1 + i}{\sqrt{3} - i}$ and $\frac{-1 + \sqrt{3}i}{2 + 2i}$ in the form $x + iy$ using two different methods.