Math 202

12 March 2010

Second Midterm

NAME (Print!): KEY

Check one: (1pm): ______
(2pm): ______

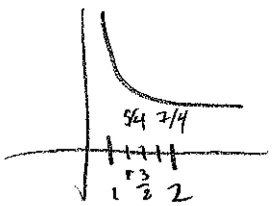
Problem	Points	Score
1	20	
2	20	
3	30	
4	20	
5	10	
Total	100	

Problem 1 (20 points): Answer the following:

(a) Using Simpson's rule with $n = 4$, approximate $\int_1^2 1/x \, dx$.

(b) For which n will Simpson's rule approximate $\int_1^2 1/x \, dx$ within 0.0001?

(a) $S_4 = \left(\frac{1}{3}\right)\left(\frac{1}{4}\right)\left(1 + 4\cdot\frac{1}{\frac{5}{4}} + 2\cdot\frac{1}{\frac{3}{2}} + 4\cdot\frac{1}{\frac{7}{4}} + 1\cdot\frac{1}{2}\right)$



$= \frac{1}{12}\left(1 + \frac{16}{5} + \frac{4}{3} + \frac{16}{7} + \frac{1}{2}\right)$

$= 0.69325\ldots$

Compare to $\int_1^2 \frac{1}{x}\,dx = \ln 2 = 0.69314\ldots$

to see that it's close.

(b) $f'(x) = -\frac{1}{x^2}$

$f''(x) = \frac{2}{x^3}$

$f'''(x) = \frac{-6}{x^4}$

$f^{(4)}(x) = \frac{24}{x^5}$

Note that $f^{(4)}(x) \geq 0$ when x in $[1,2]$, and for the same x, $f^{(5)}(x) \leq 0$.

So the function's always decreasing

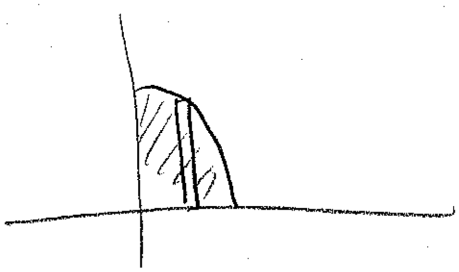
So its max happens at $x = 1$. So

$K_4 = 24$

$\text{Error}(S_N) \leq \frac{24\,(1)^5}{180N^4} \leq 0.0001$

$N \geq 7$

Problem 2 (20 points): Find the centroid of the region bounded by $y = \cos x$, $y = 0$, $x = 0$ and $x = \pi/2$.



$$\bar{x} = \frac{\int_0^{\frac{\pi}{2}} x\cos x \, dx}{\int_0^{\frac{\pi}{2}} \cos x \, dx} = \boxed{}$$

$$\bar{y} = \frac{\int_0^{\frac{\pi}{2}} \frac{1}{2}\cos^2 x \, dx}{\int_0^{\frac{\pi}{2}} \cos x \, dx} = \boxed{}$$

Problem 3 (30 points): Determine whether or not the following integrals converge or diverge. If it converges, give a reasonable upper bound for the number to which it coverges (e.g., $+\infty$ is not reasonable but 10 is).

(a) $\int_3^\infty \frac{dx}{x^2+x-6}$ $= \int_3^6 \frac{dx}{x^2+x-6} + \int_6^\infty \frac{dx}{x^2+x-6}$

Claim: $\frac{1}{x^2+x-6} < \frac{1}{x^2}$ So $\int_6^\infty \frac{dx}{x^2+x-6}$ converges.

$x^2 < x^2+x-6 \Rightarrow 6 < x$

Note: $(x^2+x-6) = (x+3)(x-2)$ so $\int_3^6 \frac{dx}{x^2+x-6}$ is a proper integral

(b) $\int_1^\infty \frac{\cos^2(x)}{1+x^2}\, dx$ $\leq \int_1^\infty \frac{1}{1+x^2}\, dx$

$= \lim_{t\to\infty} \int_1^t \frac{1}{1+x^2}\, dx$

$= \lim_{t\to\infty} \arctan t - \arctan 1$

$= \lim_{t\to\infty} \arctan t - \frac{\pi}{4}$

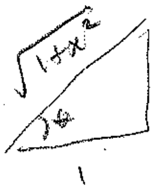
$= \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4}.$

So $\int_1^\infty \frac{\cos^2(x)}{1+x^2}\, dx \leq \frac{\pi}{4}$

Problem 4 (20 points): Find the following:

(a) The arclength of $y = \ln x$ for $1 \le x \le \sqrt{3}$.

$$\int_1^{\sqrt{3}} \sqrt{f'(x)^2+1}\,dx = \int_1^{\sqrt{3}} \sqrt{1+\frac{1}{x^2}}\,dx = \int_1^{\sqrt{3}} \frac{\sqrt{x^2+1}}{x}\,dx$$



$\tan\theta = x$

$\sec^2\theta\, d\theta = dx$

$$\int_{x=1}^{\sqrt{3}} \frac{\sqrt{\sec^2\theta}}{\tan\theta}\sec^2\theta\, d\theta = \int_{x=1}^{\sqrt{3}} \frac{\sec^3\theta}{\tan\theta}\,d\theta$$

$$= \int_{x=1}^{\sqrt{3}} \frac{\sec^2\theta\,\sec\theta}{\tan\theta}\,d\theta = \int_{x=1}^{\sqrt{3}} \frac{1+\tan^2\theta}{\tan\theta}\sec\theta\, d\theta = \int_{x=1}^{\sqrt{3}} \frac{\sec\theta}{\tan\theta} + \tan\theta\, d\theta$$

$$= \int_{x=1}^{\sqrt{3}} \csc\theta + \tan\theta\, d\theta = -\csc\theta\,\tan\theta - \ln|\cos\theta| + C$$

$$= \frac{\sqrt{1+x^2}}{x}\cdot\frac{x}{1} - \ln\frac{x}{\sqrt{1+x^2}}\Big|_1^{\sqrt{3}} = \sqrt{1+x^2} - \ln\frac{x}{\sqrt{1+x^2}} = 2 - \ln\frac{\sqrt{3}}{2} - \left(\sqrt{2} - \ln\frac{1}{\sqrt{2}}\right)$$

(b) The surface area of $y = \sqrt{x}$ rotated around the x-axis for $4 \le x \le 9$.

$$\int_a^b f(x)\sqrt{1+f'(x)^2}\,dx = \int_4^9 \sqrt{x}\sqrt{1+\left(\frac{1}{2\sqrt{x}}\right)^2} = \int_4^9 \sqrt{x\left(1+\frac{1}{4x}\right)}\,dx$$

$$= \int_4^9 \sqrt{x+\frac{1}{4}}\,dx$$

$u = x + \frac{1}{4}$

$$= \frac{2}{3}\left(x+\frac{1}{4}\right)^{3/2}$$

$$= \frac{2}{3}\left[\left(9+\frac{1}{4}\right)^{3/2} - \left(4+\frac{1}{4}\right)^{3/2}\right]$$

Problem 5 (10 points): Find the values of p for which the integral $\int_0^1 \frac{dx}{x^p}$ converges; prove your answer. For those values of p for which it converges, find the value of the integral.

$p \neq 1$

$$\lim_{a\to 0^+} \int_a^1 \frac{dx}{x^p} = \lim_{a\to 0^+} \frac{x^{-p+1}}{-p+1}\Big|_{x=a}^{1}$$

$$= \lim_{a\to 0^+} \frac{1}{-p+1} - \frac{a^{-p+1}}{-p+1}$$

$$= \frac{1}{1-p} - \lim_{a\to 0^+} \frac{a^{-p+1}}{-p+1}$$

when $-p+1>0$ $\quad a^{-p+1} \to 0$ as $a\to 0$ b/c small number to positive exponent is small

when $-p+1<0$ $\quad a^{-p+1} \to +\infty$ as $a \to 0$ b/c small number to negative exponent is big.

when $p=1$ $\lim_{a\to 0^+}\int_a^1 \frac{1}{x}\,dx = \lim_{a\to 0^+} \ln x \Big|_a^1$

$$= 0 - \lim_{a\to 0^+} \ln a$$

$$= 0 - -\infty = +\infty$$ So it diverges at $p=1$.

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