

Math 202
5 May 2008
Final Practice Exam (calculator part)

NAME (Print!): Solutions

Check one: (Dryden): _____

(Pile): _____

(Ryan): _____

Problem	Points	Score
1	30	
2	25	
3	25	
4	20	
5	20	
6	25	
7	15	
8	10	
9	10	
10	10	
11	10	
Total	200	

Problem 1: As usual, you must show your work to receive full credit.

- (1) (12 points) Find the Taylor polynomial of degree three around $a = 0$ for the function

$$f(x) = \sqrt{1-x}.$$

- (2) (10 points) Use your answer in part (1) to give approximate values to $\sqrt{\frac{1}{2}}$ and $\sqrt{0.9}$.
- (3) (8 points) Which approximation in part (2) is more accurate? Explain why, giving a rigorous mathematical justification (i.e., not just "my calculator tells me that this approximation is more accurate").

- (1) You can either take derivatives or use the first four terms of the binomial series with $p = \frac{1}{2}$ and $-x$ substituted for x . We show the latter method:

$$(1+x)^{1/2} = 1 + \frac{1}{2}x + \frac{\frac{1}{2}(\frac{1}{2}-1)}{2!}x^2 + \frac{\frac{1}{2}(\frac{1}{2}-1)(\frac{1}{2}-2)}{3!}x^3 + \dots$$

$$= 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3 + \dots$$

$$(1-x)^{1/2} = 1 + \frac{1}{2}(-x) - \frac{1}{8}(-x)^2 + \frac{1}{16}(-x)^3 + \dots = 1 - \frac{1}{2}x - \frac{1}{8}x^2 - \frac{1}{16}x^3 - \dots$$

Thus

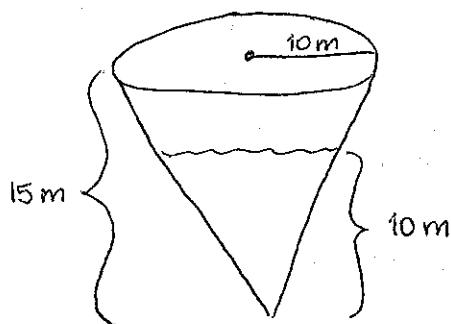
$$P_3(x) = 1 - \frac{1}{2}x - \frac{1}{8}x^2 - \frac{1}{16}x^3.$$

- (2) We have $\sqrt{\frac{1}{2}} = \sqrt{1-\frac{1}{2}}$, so $x = \frac{1}{2}$ and $P_3(\frac{1}{2}) = 1 - \frac{1}{2}(\frac{1}{2}) - \frac{1}{8}(\frac{1}{4}) - \frac{1}{16}(\frac{1}{8}) \approx .7109$.

Similarly, $\sqrt{0.9} = \sqrt{1-.1}$, so $x = 0.1$ and $P_3(0.1) = 1 - \frac{1}{2}(.1) - \frac{1}{8}(.01) - \frac{1}{16}(.001) \approx .9487$

- (3) The second approximation is more accurate. We expect this, since the polynomial is centered at $a=0$ and $x=0.1$ is closer to this center than $x=\frac{1}{2}$.

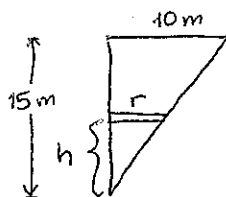
Problem 2: (25 points) A cone-shaped water reservoir is 20 m in diameter across the top and 15 m deep. If the reservoir is filled to a depth of 10 m, how much work is required to pump all the water to the top of the reservoir? (The density of water is 1000 kg/m^3 .)



We slice the water horizontally into circular slabs, with each slab having thickness Δh . We let h denote the distance from the bottom of the reservoir.



Then $\text{vol}(\text{slab}) = \pi r^2 \Delta h$, and we need to find r in terms of h . We use similar triangles:



$$\frac{10}{15} = \frac{r}{h} \Rightarrow r = \frac{2}{3}h$$

$$\text{So } \text{vol}(\text{slab}) = \pi \left(\frac{2}{3}h\right)^2 \Delta h = \frac{4}{9} \pi h^2 \Delta h,$$

$$\text{force}(\text{slab}) = \underbrace{\left(\frac{4}{9} \pi h^2 \Delta h\right)}_{\text{vol.}} \underbrace{(1000)}_{\text{density}} \underbrace{(9.8)}_{\text{gravity}},$$

$$\text{work to move slab} = \underbrace{\left(\frac{4}{9} \pi \cdot 9800 h^2 \Delta h\right)}_{\text{force}} \underbrace{(15-h)}_{\text{distance}}.$$

Thus the total work to pump all the water to the top is

$$\text{Total work} = \int_0^{10} \left(\frac{4}{9} \pi \cdot 9800 h^2\right) (15-h) dh$$

$$= \frac{4}{9} \pi \cdot 9800 \int_0^{10} (15h^2 - h^3) dh$$

$$= \frac{4}{9} \pi \cdot 9800 \left[5h^3 - \frac{h^4}{4} \right]_0^{10}$$

$$= \frac{4}{9} \pi \cdot 9800 [5000 - 2500 - 0]$$

$$= \frac{4}{9} \pi \cdot 9800 \cdot 2500 \approx 3.42 \times 10^7 \text{ joules.}$$

Problem 3: In this problem, you will be working with the differential equation

$$y' = y + xy.$$

We are interested in the solution that satisfies the initial condition $y(0) = 1$.

- (1) (10 points) Use Euler's Method with step size 0.1 to estimate $y(0.5)$.
- (2) (5 points) Is your estimate an overestimate or an underestimate? Give justification for your answer using the slope field. You don't need to draw the slope field, but you should graph it on your calculator and describe it as necessary to support your justification.
- (3) (10 points) Find the exact solution to this initial-value problem.

(1) Step	x	approximate y-value	$\Delta y = \text{slope} \cdot \Delta x = (y+xy)(0.1)$
0	0	1	$(1+0)(0.1) = .1$
1	0.1	1.1	$(1.1 + (0.1)(1.1))(0.1) \approx .121$
2	0.2	1.22	$(1.22 + (0.2)(1.22))(0.1) \approx .146$
3	0.3	1.37	$(1.37 + (0.3)(1.37))(0.1) \approx .178$
4	0.4	1.55	$(1.55 + (0.4)(1.55))(0.1) \approx .217$
5	0.5	1.77	

Using Euler's Method we estimate $y(0.5) \approx 1.77$.

(2) My estimate is an underestimate. The slope field shows that the solution curves are concave up, so Euler's method will take us below the actual value.

(3) We use separation of variables:

$$\frac{dy}{dx} = y(1+x)$$

$$\int \frac{dy}{y} = \int (1+x) dx$$

$$\ln|y| = x + \frac{x^2}{2} + C$$

$$|y| = e^{x + x^2/2 + C}$$

$$y = Ae^x e^{x^2/2} \quad (A = \pm e^C)$$

Using the initial condition $y(0) = 1$ gives

$$1 = Ae^0 e^0 = A,$$

so the exact soln. is $y = e^x e^{x^2/2}$.

NAME (Print!): Solutions

Check one: (Dryden): _____

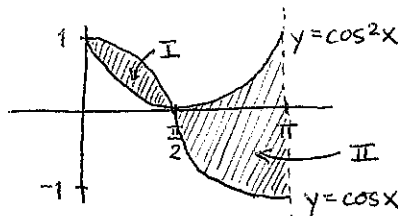
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Problem 4: (20 points) Find the area of the region bounded by the curves $y = \cos x$ and $y = \cos^2 x$ between $x = 0$ and $x = \pi$.

We begin by graphing the region:

We split the region into 2 pieces, labeled I and II.



$$\text{Area (I)} = \int_0^{\pi/2} (\cos x - \cos^2 x) dx$$

$$\text{Area (II)} = \int_{\pi/2}^{\pi} (\cos^2 x - \cos x) dx$$

So $\text{Area}(\text{region}) = \text{Area (I)} + \text{Area (II)} = \int_0^{\pi/2} (\cos x - \cos^2 x) dx + \int_{\pi/2}^{\pi} (\cos^2 x - \cos x) dx.$

Note that we need to integrate $\cos^2 x$ twice! We can either use the identity

$$\cos^2 x = \frac{1 + \cos 2x}{2} \text{ or do integration by parts, i.e.,}$$

$$\int \cos^2 x dx = \int \frac{1 + \cos 2x}{2} dx = \int \left(\frac{1}{2} + \frac{1}{2} \cos 2x \right) dx = \frac{1}{2} x + \frac{1}{4} \sin 2x + C = \frac{1}{2} x + \frac{1}{2} \sin x \cos x + C$$

$$\int \cos^2 x dx = \cos x \sin x + \int \sin^2 x dx = \cos x \sin x + \int (1 - \cos^2 x) dx = \cos x \sin x + x - \int \cos^2 x dx$$

$$u = \cos x \quad v' = \cos x$$

$$u' = -\sin x \quad v = \sin x$$

$$\Rightarrow 2 \int \cos^2 x dx = x + \cos x \sin x + C$$

$$\int \cos^2 x dx = \frac{1}{2} x + \frac{1}{2} \cos x \sin x + C$$

Thus

$$\begin{aligned} \text{Area}(\text{region}) &= \int_0^{\pi/2} \cos x dx - \int_0^{\pi/2} \cos^2 x dx + \int_{\pi/2}^{\pi} \cos^2 x dx - \int_{\pi/2}^{\pi} \cos x dx \\ &= (\sin x \Big|_0^{\pi/2}) - \left(\frac{1}{2} x + \frac{1}{2} \sin x \cos x \Big|_0^{\pi/2} \right) + \left(\frac{1}{2} x + \frac{1}{2} \sin x \cos x \Big|_{\pi/2}^{\pi} \right) - (\sin x \Big|_{\pi/2}^{\pi}) \\ &= (1 - 0) - \left(\frac{1}{2} \cdot \frac{\pi}{2} + \frac{1}{2} \cdot 0 - 0 \right) + \left(\frac{1}{2} \cdot \pi + \frac{1}{2} \cdot 0 - \frac{1}{2} \cdot \frac{\pi}{2} - \frac{1}{2} \cdot 0 \right) - (0 - 1) \\ &= 1 - \frac{\pi}{4} + \frac{\pi}{2} - \frac{\pi}{4} + 1 = \boxed{2} \end{aligned}$$

Problem 5: (20 points) Determine whether the improper integral given below converges. If it does, evaluate it.

$$\int_1^{\infty} \frac{\tan^{-1} x}{x^3} dx$$

We see that this integral is improper at the upper limit, and only there.

We deal with the indefinite integral first:

$$\int \frac{\tan^{-1} x}{x^3} dx = -\frac{\tan^{-1} x}{2x^2} + \frac{1}{2} \int \frac{1}{x^2(1+x^2)} dx$$

$$u = \tan^{-1} x \quad v' = x^{-3}$$

$$u' = \frac{1}{1+x^2} \quad v = -\frac{1}{2x^2}$$

use partial fractions!

$$\frac{1}{x^2(1+x^2)} = \frac{A}{x} + \frac{B}{x^2} + \frac{Cx+D}{1+x^2}$$

multiply both sides by $x^2(1+x^2)$

$$1 = Ax(1+x^2) + B(1+x^2) + (Cx+D)x^2$$

$$1 = Ax + Ax^3 + B + Bx^2 + Cx^3 + Dx^2 = x^3(A+C) + x^2(B+D) + Ax + B$$

$$\text{So } A+C=0 \quad B+D=0 \quad A=0 \quad B=1$$

$$\Rightarrow C=0 \quad \Rightarrow D=-1$$

Thus

$$\int \frac{1}{x^2(1+x^2)} dx = \int \left(\frac{1}{x^2} - \frac{1}{1+x^2} \right) dx = -\frac{1}{x} - \tan^{-1} x + C$$

Now let's go back to the original integral and put the pieces together:

$$\int_1^{\infty} \frac{\tan^{-1} x}{x^3} dx = \lim_{b \rightarrow \infty} \int_1^b \frac{\tan^{-1} x}{x^3} dx$$

$$= \lim_{b \rightarrow \infty} \left(-\frac{\tan^{-1} x}{2x^2} + \frac{1}{2} \left(-\frac{1}{x} - \tan^{-1} x \right) \right) \Big|_1^b$$

$$= \lim_{b \rightarrow \infty} \left(-\frac{\tan^{-1} b}{2b^2} - \frac{1}{2b} - \frac{1}{2} \tan^{-1} b + \frac{\tan^{-1} 1}{2} + \frac{1}{2} + \frac{1}{2} \tan^{-1} 1 \right)$$

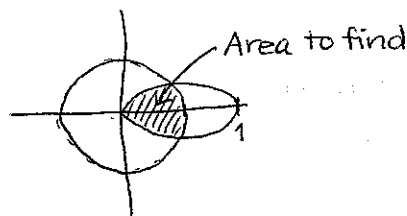
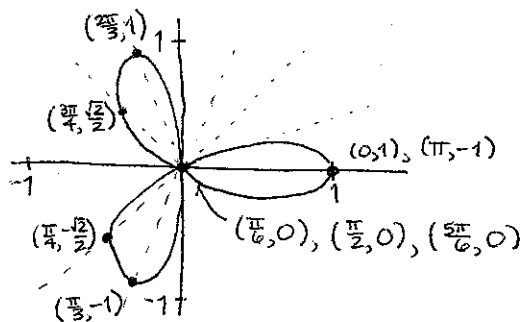
$$= 0 - 0 - \frac{1}{2} \cdot \frac{\pi}{2} + \frac{1}{2} \cdot \frac{\pi}{4} + \frac{1}{2} + \frac{1}{2} \cdot \frac{\pi}{4}$$

$$= \frac{1}{2} + \frac{\pi}{4} - \frac{\pi}{4} = \boxed{\frac{1}{2}}$$

Thus the integral converges to $\frac{1}{2}$.

Problem 6: (25 points) Complete the following table and sketch the graph of $r = \cos 3\theta$. Please label the nine points on the curve. Then write an integral representing the area of the region enclosed by both the circle $r = 1/2$ and one petal of the curve $r = \cos 3\theta$. You do not need to evaluate your integral.

θ	0	$\pi/6$	$\pi/4$	$\pi/3$	$\pi/2$	$2\pi/3$	$3\pi/4$	$5\pi/6$	π
r	1	0	$-\frac{\sqrt{2}}{2}$	-1	0	1	$\frac{\sqrt{2}}{2}$	0	-1



We have

$$\text{Area (petal)} = 2 \int_0^{\pi/6} \frac{1}{2} (\cos 3\theta)^2 d\theta = \int_0^{\pi/6} \cos^2 3\theta d\theta$$

$$\begin{aligned} \text{Area inside petal, but outside circle} &= 2 \int_0^{\pi/4} \frac{1}{2} (\cos 3\theta)^2 - \frac{1}{2} \left(\frac{1}{2}\right)^2 d\theta \\ &= \int_0^{\pi/4} (\cos^2 3\theta - \frac{1}{4}) d\theta \end{aligned}$$

where we found the upper bound as follows:

$$\cos 3\theta = \frac{1}{2} \Rightarrow 3\theta = \pm \frac{\pi}{3} \Rightarrow \theta = \pm \frac{\pi}{9}$$

Thus the area we want to find is given by

$$A = \int_0^{\pi/6} \cos^2 3\theta d\theta - \int_0^{\pi/4} (\cos^2 3\theta - \frac{1}{4}) d\theta.$$

Problem 7: (15 points) Determine whether the series given below converges conditionally, converges absolutely or diverges. Be sure to name any test(s) you use and verify that the series satisfies all necessary hypotheses.

$$\sum_{n=1}^{\infty} \frac{2^{3n}}{n^2 5^n}$$

We note that
$$\sum_{n=1}^{\infty} \frac{2^{3n}}{n^2 5^n} = \sum_{n=1}^{\infty} \frac{(2^3)^n}{n^2 5^n} = \sum_{n=1}^{\infty} \frac{8^n}{n^2 5^n} = \sum_{n=1}^{\infty} \left(\frac{8}{5}\right)^n \cdot \frac{1}{n^2}.$$

Since $\left(\frac{8}{5}\right)^n$ grows larger as n increases, we suspect that this series diverges. We confirm this via the Ratio Test:

$$\lim_{n \rightarrow \infty} \left| \frac{8^{n+1}}{5^{n+1} (n+1)^2} \cdot \frac{5^n n^2}{8^n} \right| = \frac{8}{5} \lim_{n \rightarrow \infty} \frac{n^2}{n^2 + 2n + 1} = \frac{8}{5} \cdot 1 = \frac{8}{5} > 1$$

Since the result is bigger than 1, our series diverges.

Problem 8: (10 points) Compute

$$\left(\frac{1}{2} + \frac{\sqrt{3}}{2}i\right)^{15}$$

and express your answer in the form $a + bi$ for real numbers a and b .

We have $r = \sqrt{\left(\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2} = \sqrt{\frac{1}{4} + \frac{3}{4}} = \sqrt{1} = 1$. Now

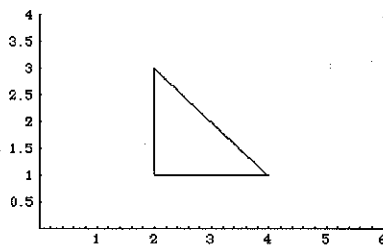
$$\cos \theta = \frac{\frac{1}{2}}{1} = \frac{1}{2}, \quad \sin \theta = \frac{\frac{\sqrt{3}}{2}}{1} = \frac{\sqrt{3}}{2} \Rightarrow \theta = \frac{\pi}{3},$$

so $\frac{1}{2} + \frac{\sqrt{3}}{2}i = 1 \cdot e^{i\pi/3}$. Thus

$$\left(\frac{1}{2} + \frac{\sqrt{3}}{2}i\right)^{15} = \left(e^{i\pi/3}\right)^{15} = e^{i5\pi} = e^{5\pi i} = \cos 5\pi + i \sin 5\pi = -1 + 0i.$$

Hence $\left(\frac{1}{2} + \frac{\sqrt{3}}{2}i\right)^{15} = -1$.

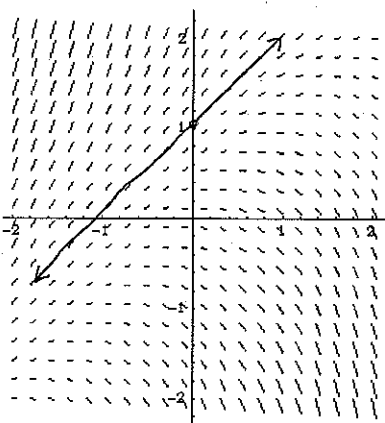
Problem 9: (10 points) Imagine rotating the triangular region shown below around the x -axis, the y -axis and the line $x = 6$. Select the list which gives the three volumes in increasing order.



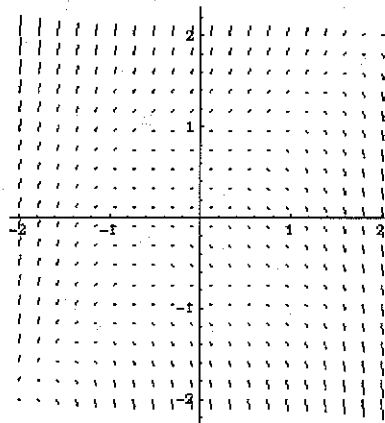
- (a) x -axis, y -axis, $x = 6$
 (b) x -axis, $x = 6$, y -axis,
 (c) y -axis, x -axis, $x = 6$
 (d) y -axis, $x = 6$, x -axis
 (e) $x = 6$, x -axis, y -axis
 (f) $x = 6$, y -axis, x -axis

The volume depends on the distance from the axis of rotation to the triangle, since volume increases with "radius." The triangle is 1 unit from the x -axis, 2 units from the y -axis, and at least 2 units from $x = 6$.

Problem 10: (10 points) One of the two pictures below corresponds to the differential equation $y' = y - x$ and the other to $y' = y^3 - x^3$. Identify which is which. Also, on the first one, sketch the solution that satisfies the initial condition $y(0) = 1$.



$$y' = y - x$$



$$y' = y^3 - x^3$$

(this picture has steeper slopes as $|x|$ increases)

Problem 11: (10 points) If $\sum_{n=0}^{\infty} c_n(4)^n$ converges, then

- (a) $\sum_{n=0}^{\infty} c_n(-4)^n$ must converge.
- (b) $\sum_{n=0}^{\infty} c_n(-4)^n$ may converge.
- (c) $\sum_{n=0}^{\infty} c_n(-4)^n$ must diverge.

Apply the Ratio Test to $\sum_{n=0}^{\infty} c_n(-4)^n$ and get $4 \lim_{n \rightarrow \infty} \left| \frac{c_{n+1}}{c_n} \right|$. But this is the same expression we would get from applying the Ratio Test to $\sum_{n=0}^{\infty} c_n(4)^n$, so if that series converges, so must our new series.