Math 201

October 26, 2010
Second (Sample) Midterm

NAME (Print!): KEY

Check one: (12pm): ______
(1pm): ______

Problem	Points	Score
1	20	
2	20	
3	30	
4	20	
5	10	
Total	100	

Problem 1 (20 points): Newton's Law of Gravitation states that the magnitude F of the force exerted by a body of mass m on a body of mass M is

$$F = \frac{GMm}{r^2}$$

where G is the gravitational constant and r is the distance between the two bodies.

(a) Find $\frac{dF}{dr}$ and explain its meaning. What does the minus sign indicate?

(b) Suppose that it is known Earth attracts an object with a force that decreases at the rate of 2 N/km when $r = 20,000$ km. How fast does this force change when $r = 10,000$ km.

$$\frac{dF}{dr} = -2\,\frac{GMm}{r^3}$$

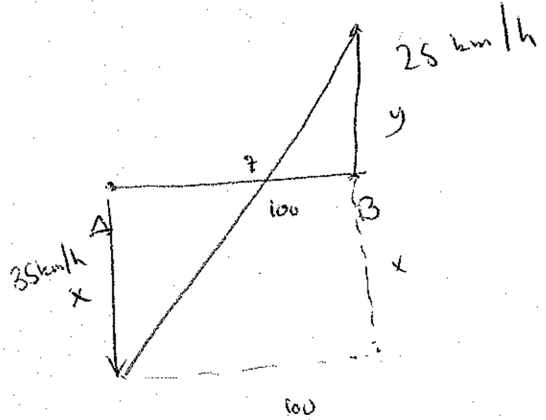
This means that as I increase the radius the Force decreases in a way that ~~deeper~~ is inversely proportional with the radius cubed.

$$\left.\frac{dF}{dr}\right|_{r=20000} = -\frac{2GMm}{(20000)^3} = -2$$

$$\Rightarrow GMm = (20000)^3$$

$$\Rightarrow \left.\frac{dF}{dr}\right|_{r=10000} = \frac{-2\,(20000)^3}{10000^3} = -16 \text{ N/km.}$$

Problem 2 (20 points): At noon ship A is 100 km west of ship B. Ship A is sailing south at 35 km/h and Ship B is sailing north at 25 km/h. How fast is the distance between the ships changing at 4:00 pm.



$\frac{dx}{dt} = 35$

$\frac{dy}{dt} = 25$

want $\frac{dz}{dt}$

$z^2 = 100^2 + (x+y)^2$

$2z\frac{dz}{dt} = 2(x+y)\left(\frac{dx}{dt} + \frac{dy}{dt}\right)$

after 4 hours, $x = 140$

$y = 100$

$z = \sqrt{100^2 + 140^2}$

$$\boxed{\frac{dz}{dt} = \frac{2(140+100)(35+25)}{2\sqrt{100^2+140^2}}}$$

Problem 3 (30 points): Find $\frac{dy}{dx}$ for each of the following:

(a) $\tan(x-y) = \frac{y}{1+x^2}$

$$\frac{d}{dx}\tan(x-y) = \frac{d}{dx}\left(y(1+x^2)^{-1}\right)$$

$$\sec^2(x-y)\left(-\frac{dy}{dx}\right) = \frac{dy}{dx}(1+x^2)^{-1} + y\,(1+x^2)^{-2}(2x)$$

$$\Rightarrow \quad -\frac{2xy}{(1+x^2)^2}\cdot\frac{1}{(1+x^2)^{-1}+\sec^2(x-y)} = \frac{dy}{dx}$$

(b) $y = 2^{3^{x^2}}$

$$\frac{dy}{dx} = 2^{3^{x^2}}\cdot(3^{x^2})'\ln 2$$

$$= 2^{3^{x^2}}(\ln 2)(\ln 3)\,3^{x^2}(x^2)'$$

$$= \ln 2\ln 3\;2^{3^{x^2}}3^{x^2}(2x)$$

(c) $y = x^{e^x}$

$$\ln y = \ln x^{e^x}$$

$$\frac{1}{y}\frac{dy}{dx} = \frac{d}{dx}e^x\ln x$$

$$\frac{1}{y}\frac{dy}{dx} = e^x\ln x + \frac{e^x}{x}$$

$$\frac{dy}{dx} = x^{e^x}\left(e^x\ln x + \frac{e^x}{x}\right)$$

Problem 4 (20 points): Prove the following differentiation rules:

(a) Using the limit definition of the derivative, prove $\frac{d}{dx}\sqrt{x} = \frac{1}{2\sqrt{x}}$.

$$\lim_{h\to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h} = \lim_{h\to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h} \frac{(\sqrt{x+h} + \sqrt{x})}{(\sqrt{x+h} + \sqrt{x})}$$

$$= \lim_{h\to 0} \frac{x+h-x}{h(\sqrt{x+h} + \sqrt{x})}$$

$$= \lim_{h\to 0} \frac{h}{h(\sqrt{x+h} + \sqrt{x})} \qquad h \neq 0$$

$$= \lim_{h\to 0} \frac{1}{\sqrt{x+h} + \sqrt{x}} \qquad \frac{1}{\sqrt{x+h} + \sqrt{x}} \text{ is continuous}$$

$$= \frac{1}{\sqrt{x} + \sqrt{x}} = \frac{1}{2\sqrt{x}}$$

(b) Show that for any real number n we have $\frac{d}{dx}x^n = nx^{n-1}$.

$$y = x^n$$

$$\ln y = n \ln x$$

$$\frac{1}{y}\frac{dy}{dx} = \frac{n}{x}$$

$$\frac{dy}{dx} = y\frac{n}{x}$$

$$= x^n \cdot \frac{n}{x}$$

$$= nx^{n-1}$$

Problem 5 (10 points): Find an equation of the tangent line to the curve $y = 4\sin^2(x)$ at the point $(\pi/6, 1)$.

$$\frac{dy}{dx} = 8\sin x \cos x$$

$$\left.\frac{dy}{dx}\right|_{\frac{\pi}{6}} = 8 \cdot \frac{1}{2} \cdot \frac{\sqrt{3}}{2}$$

$$= 2\sqrt{3}$$

Eqn of the tangent line:

$$\boxed{y - 1 = 2\sqrt{3}\left(x - \frac{\pi}{6}\right)}$$