Name:

MATH 192 Final Exam Hand Portion

Show all work to receive credit. Calculators are not permitted on this portion of the exam. You are encouraged to turn in this portion of the exam and receive the calculator portion after 1 hour; you are strongly encouraged to make the switch after 1 hour and 15 minutes; you are required to turn in this portion of the exam, and receive the calculator portion, no later than 1 hour and 30 minutes into the examination.

1. Compute as requested:

a) Find the equation of the tangent line to the graph of $y = 3x^2 - 2x + 4$ at $x = 2$.

$y' = 6x - 2$

$m = y'\big|_{x=2} = 12 - 2 = 10$

$y\big|_{x=2} = 12 - 4 + 4 = 12$

$y - 12 = 10(x - 2)$

$y = 10x - 8$

b) Find $\frac{d}{dx}\left(2x + \sqrt{x} - e^x\right)$.

$= \frac{d}{dx}\left(2x + x^{1/2} - e^x\right)$

$= 2 + \frac{1}{2}x^{-1/2} - e^x$

$= 2 + \frac{1}{2\sqrt{x}} - e^x$

c) Find $\frac{du}{dx}$ if $u = (x+1)^{12}\ln(x)$.

$\frac{du}{dx} = 12(x+1)^{11}\ln x + \frac{(x+1)^{12}}{x}$

d) Find $f'(x)$ if $f(x) = e^{3x^2} + \sqrt[3]{x^2+1}$. $= e^{3x^2} + (x^2+1)^{1/3}$

$$f'(x) = e^{3x^2}(6x) + \frac{1}{3}(x^2+1)^{-2/3}(2x)$$

$$= 6xe^{3x^2} + \frac{2x}{3\sqrt[3]{(x^2+1)^2}}$$

e) Find $f(x)$ if $f'(x) = -3e^{2x}$ and $f(0) = 1$.

$$f(x) = \int -3e^{2x}\,dx = -3\,\frac{1}{2}\,e^{2x} + C = -\frac{3}{2}e^{2x} + C$$

$$f(0) = -\frac{3}{2} + C = 1 \implies C = 1 + \frac{3}{2} = \frac{5}{2}$$

$$f(x) = -\frac{3}{2}e^{2x} + \frac{5}{2}$$

f) Compute $\int_1^3 x^3 - 2x^2 + 5\,dx$.

$$= \left(\frac{x^4}{4} - 2\,\frac{x^3}{3} + 5x\right)\Big|_1^3$$

$$= \frac{81}{4} - 18 + 15 - \left(\frac{1}{4} - \frac{2}{3} + 5\right)$$

$$= \frac{80}{4} - 8 + \frac{2}{3} = 12 + \frac{2}{3} = \frac{38}{3}$$

g) Compute $\int \frac{1}{x} + (1+3x)^2\,dx$. $= \int \frac{1}{x} + 1 + 6x + 9x^2\,dx$

$$= \ln|x| + x + 3x^2 + 3x^3 + C$$

2. Find the average value of $4x^2$ on the interval $[1, 5]$.

$$f_{av} = \frac{1}{5-1}\int_1^5 4x^2\,dx = \frac{1}{4}\,\frac{4}{3}x^3\Big|_1^5$$

$$= \frac{1}{3}(125-1) = \frac{124}{3}$$

3. Find the area between the graphs of $f(x) = 2x + 1$ and $g(x) = x^2 + 1$.

intersection: $2x+1 = x^2+1$

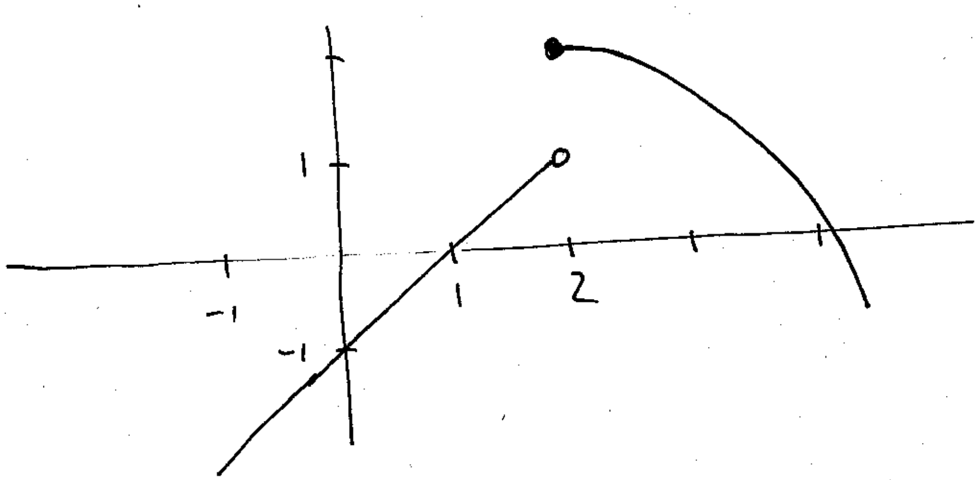
$x^2 - 2x = x(x-2) = 0$ $\quad x=0,\ x=2$

$$\text{Area} = \left| \int_0^2 x^2+1 - (2x+1)\,dx \right|$$

$$= \left| \int_0^2 x^2 - 2x\,dx \right| = \left| \left(\frac{x^3}{3} - x^2\right)\Big|_0^2 \right|$$

$$= \left| \frac{8}{3} - 4 \right| = \left| \frac{8-12}{3} \right| = \frac{4}{3}$$

4. Sketch, on the axes provided, a clear graph of a function that is defined at $x = 2$, has a left hand limit at $x = 2$, and has a right hand limit at $x = 2$, but has no limit at $x = 2$.



5. Sketch the graph of f defined by $f(x) = 5x^3 - 3x^5$. List the coordinates of where extrema and points of inflection occur, and label them on your graph. State where the function is increasing or decreasing, as well as where it is concave up or concave down.

$f(x) = 5x^3 - 3x^5 = x^3(5 - 3x^2)$

$f(x) = 0 \qquad 5 - 3x^2 = 0$ or $x = 0 \implies 3x^2 = 5 \qquad x = \pm\sqrt{5/3},\ 0$

$f'(x) = 15x^2 - 15x^4 = 15x^2(1 - x^2)$

$f'(x) = 0 \qquad 1 - x^2 = 0$ or $x = 0 \qquad x = 0,\ 1,\ -1$

$f''(x) = 30x - 60x^3 = 30x(1 - 2x^2)$

$f''(x) = 0 \qquad 1 - 2x^2 = 0$ or $1 - 2x^2 = 0 \qquad x = 0,\ \pm\sqrt{1/2}$

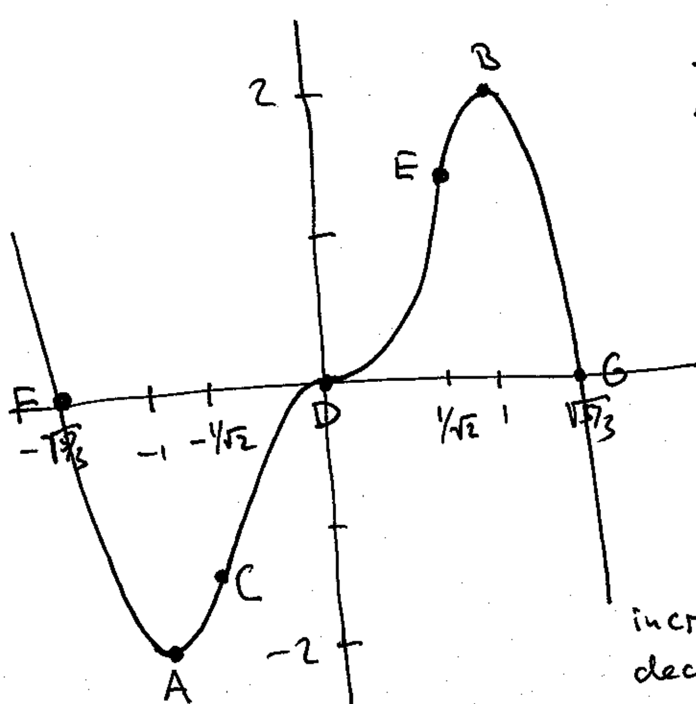
decreasing | increasing | increasing | decreasing

$-\sqrt{5/3} \quad -1 \quad -1/\sqrt{2} \quad 0 \quad 1/\sqrt{2} \quad 1 \quad \sqrt{5/3}$

y' — 0 + 0 + 0 —

y'' + 0 — 0 + 0 —

$f(-\frac{1}{\sqrt{2}}) = -5\frac{1}{2\sqrt{2}} + 3\frac{1}{4\sqrt{2}} = \frac{3-10}{4\sqrt{2}} = -\frac{7}{4\sqrt{2}}$



	x	y	
A	-1	-2	local min
B	1	2	local max
C	$-1/\sqrt{2}$	$\frac{-7}{4\sqrt{2}}$	inflection point
D	0	0	inflection point, intercept
E	$\frac{1}{\sqrt{2}}$	$+\frac{7}{4\sqrt{2}}$	inflection point
F	$-\sqrt{5/3}$	0	x-intercept
G	$\sqrt{5/3}$	0	x-intercept

increasing: $(-1, 1)$

decreasing: $(-\infty, -1) \cup (1, \infty)$

concave up: $(-\infty, -1/\sqrt{2}) \cup (0, 1/\sqrt{2})$

concave down: $(-\frac{1}{\sqrt{2}}, 0) \cup (\frac{1}{\sqrt{2}}, \infty)$

6. Find the absolute maximum and minimum values of f defined by $f(x) = x^3 - 6x^2 + 10$ on the interval $[-3, 5]$ and indicate the x-values at which they occur.

$f'(x) = 3x^2 - 12x = 3x(x-4)$, critical points at $x = 0, 4$

critical points and endpoints

x	y
-3	-71
0	10
4	-22
5	-15

$f(-3) = -27 - 54 + 10 = -71$
$f(0) = 10$
$f(4) = 64 - 96 + 10 = -22$
$f(5) = 125 - 150 + 10 = -15$

The absolute maximum is 10 , it occurs at $x = 0$
The absolute minimum is -71 , it occurs at $x = -3$

Name:

MATH 192 Final Exam Calculator Portion

Show all work to receive credit. Calculators are permitted on this portion of the exam, but please try to make clear what is done by calculator. Note that the last page of this examination is a table of the values of the standard normal variable.

7. A company producing widgets finds that if it produces x widgets it should sell them at the price \$ $215x - x^2$ *per widget.* The company has fixed costs of \$ 5,000 ; other than these, if it produces x widgets the cost of production is \$ $x^2 - 100x$ *per widget.* Determine, by methods of calculus, the number of widgets the company should produce for maximum profit, and justify your answer by showing that it is indeed the maximum.

Revenue $R(x) = (215x - x^2)x = 215x^2 - x^3$

Cost $C(x) = 5000 + (x^2 - 100x)x = 5000 + x^3 - 100x^2$

$P(x) = R(x) - C(x) \quad , \; x > 0$

$= 215x^2 - x^3 - 5000 - x^3 + 100x^2$

$= -2x^3 + 315x^2 - 5000$

$P'(x) = -6x^2 + 630x = 6x(-x + 105)$

$P'(x) = 0 \Rightarrow x = 0, 105$

$P''(x) = -12x + 630$

$P''(105) = -12(105) + 630 < 0$

Hence we have a relative max at $x = 105$ and there is no other one in the domain. Thus we have an abs. max.

The production of 105 widgets yields a maximum profit.

8. A company finds that the demand function for its product is $q = D(x) = 967 - 25x$, where q is the number of units sold when the price per unit, in cents, is x.

a) Find the elasticity.

$$E = -\frac{x\,D'(x)}{D(x)} = -\frac{x(-25)}{967-25x} = \frac{25x}{967-25x}$$

b) At what price(s) is the elasticity of demand equal to 1?

$$E = 1 = \frac{25x}{967-25x} \Rightarrow 25x = 967 - 25x$$

$$50x = 967, \quad x = \frac{967}{50} = 19.34 \text{ cents}$$

c) At what prices is the elasticity of demand elastic? Interpret, in practical terms, what this means for a small increase in price.

The demand is elastic if $E > 1$ which is for prices in the range $\left(\frac{967}{50}, \frac{967}{25}\right)$

A small increase in price will decrease the revenue in that price range.

d) At what price is the revenue a maximum?

The revenue is a maximum if $E = 1$, that is when the price is $\frac{967}{50}$ cents.

9. a) If at the beginning of the year 2000, \$ 1,000 was placed in a bank account yielding 5% annual interest compounded continuously, what is the balance at the beginning of 2002?

$$\text{Amount} = 1000\, e^{2(.05)}$$

$$= 1000\, e^{.1}$$

$$= \$\ 1105.17$$

b) What is the doubling time for the investment of part a)?

Find T such that $1000\, e^{.05T} = 2000$

$$e^{.05T} = 2$$

$$.05T = \ln 2$$

$$T = \frac{\ln 2}{.05} \approx 13.86 \text{ years}$$

c) Consider a different investment: at the beginning of the year 2000, \$ 1,000 was placed in an account paying interest compounded continuously, and at the beginning of 2003 the balance in the account was \$1,250 . Find a formula for the value of this investment at time t, where t is measured in years since the beginning of 2000.

$P(t)$ = value after t years.

$$= 1000\, e^{kt}$$

$$P(3) = 1250 = 1000\, e^{3k}$$

$$e^{3k} = 1.25$$

$$3k = \ln 1.25$$

$$k = \frac{\ln 1.25}{3}$$

$$P(t) = 1000\, e^{\frac{\ln 1.25}{3} t}$$

9. (continued) d) Consider a third investment: what is the future value of a continuous money flow at the rate $R(t) = 300$ (dollars), at annual interest rate 5%, compounded continuously over 4 years?

Future value $= e^{(.05)4} \int_0^4 300\, e^{-.05t}\, dt$

$\cong \$\ 1328.42$

e) Find the future values of the investments in a), c), and d) at year 4 (indicate clearly which is which, and we assume the third investment began in year 2000). Which is preferable from this point of view?

Future values at 4 years for ~~invest~~

investment a): $1000\, e^{(.05)4} = \$\ 1221.40$

c): $1000\, e^{\frac{\ln 1.25}{3} 4} = \$\ 1346.52$

d) $\$\ 1328.42$

Investment c) gives the biggest future value in 4 years, so is preferable.

9. (continued) f) If the annual interest rate is 5% compounded continuously, each of the payouts in year 2004 had a present value in the year 2000. Compute these present values for investments a), c), and d) (indicate clearly which is which). Which investment is preferable from the point of view of present value in the year 2000?

present value is Future $\cdot e^{-(.05)4}$

For a): present value $= 1221.40\, e^{-.05(4)}$

$= \$\ \cancel{999}\ 1000.00$

c) present value $= 1346.52\, e^{-.05(4)}$

$= \$\ 1102.44$

d) present value $= 1328.42\, e^{-.05(4)}$

$= \$\ 1087.62$

For a) you have to put up the smallest amount of money so in some sense this is preferable.

g) One of the present values you computed in f) is, in 20-20 hindsight, completely obvious. Which, and why?

a) The future value was calculated based on an investment of \$1000 with the same interest rate and method of compounding as the reverse calculation of the present value.

10. The duration of a cash register transaction, in minutes, at a local store is a continuous random variable with a probability density function of the form $f(t) = \frac{3}{760}t^2 - \frac{27}{380}t + \frac{123}{380}$ over the interval $[0, 10]$.

a) What is the probability that a cash register transaction lasts less than 2 minutes?

X = length of cash register transaction

$$P(X < 2) = \int_0^2 \frac{3}{760}t^2 - \frac{27}{380}t + \frac{123}{380}\,dt = \frac{49}{95}$$

b) What is the probability that a cash register transaction lasts longer than 8 minutes?

$$P(X > 8) = \int_8^{10} \frac{3}{760}t^2 - \frac{27}{380}t + \frac{123}{380}\,dt = \frac{1}{95}$$

c) What is the expected value of the duration of a cash register transaction?

$$EX = \int_0^{10} t\left(\frac{3}{760}t^2 - \frac{27}{380}t + \frac{123}{380}\right)dt =$$

$$= \frac{45}{19} \simeq 2.37 \text{ minutes.}$$

11. The score s on a certain test in a high school is normally distributed with mean $\mu = 13$ and standard deviation $\sigma = 3$.

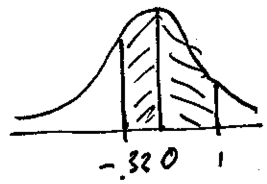
a) Find the probability that s falls between 12 and 16; that is, find $P(12 \leq s \leq 16)$.

$$P(12 \leq s \leq 16) = P\left(\frac{12-13}{3} \leq z \leq \frac{16-13}{3}\right)$$

$$= P(-0.33 \leq z \leq 1.00)$$

$$= P(0 \leq z \leq .33) + P(0 \leq z \leq 1.00)$$

$$= .1293 + .3413 = .4706$$



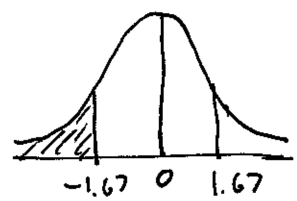
b) Find the expected value of a score.

$$E(s) = \mu = 13$$

c) The school loses some state revenue for each student scoring below 8. For what proportion of the students does the school lose the revenue?

$$P(s < 8) = P\left(z < \frac{8-13}{3}\right)$$

$$= P(z < -1.67)$$



$$= .5 - P(0 < z < 1.67)$$

$$= .5 - .4525$$

$$= .0475$$

They loose the revenue for 4.75% of the students.