Equality of Stanley Symmetric Functions

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Formal Definition

For a permutation w, we define the corresponding Stanley symmetric function in m variables as:

$$F_w(x_1, \dots, x_m) = \sum_{a \in R(w)} Q_{D(a),\ l(w)}(x_1, \dots, x_m),$$

where $R(w)$ is the set of reduced words of w, $D(a)$ is the descent set of a, $l(w)$ is the length (number of inversions) of w, and Q's are fundamental quasisymmetric functions.

Goal

Determine when $F_{w_1} = F_{w_2}$.

Equality Condition

$F_{w_1} = F_{w_2}$ if and only if $D(R(w_1)) = D(R(w_2))$, where $D(R(w)) = \{D(a) \mid a \in R(w)\}$.

This allows us to only look at the descent sets of the reduced words to check for Stanley symmetric function equality.

Example

Consider $w_1 = 4132 \in S_4$. We have:

$$R(4132) = \{\sigma_3\sigma_2\sigma_1\sigma_3, \sigma_3\sigma_2\sigma_3\sigma_1, \sigma_2\sigma_3\sigma_2\sigma_1\}$$
$$D(R(4132)) = \{(1,2), (1,3), (2,3)\}$$
$$F_{4132} = Q_{(1,2),\ 4} + Q_{(1,3),\ 4} + Q_{(2,3),\ 4}$$

Now consider $25134 \in S_5$. We have:

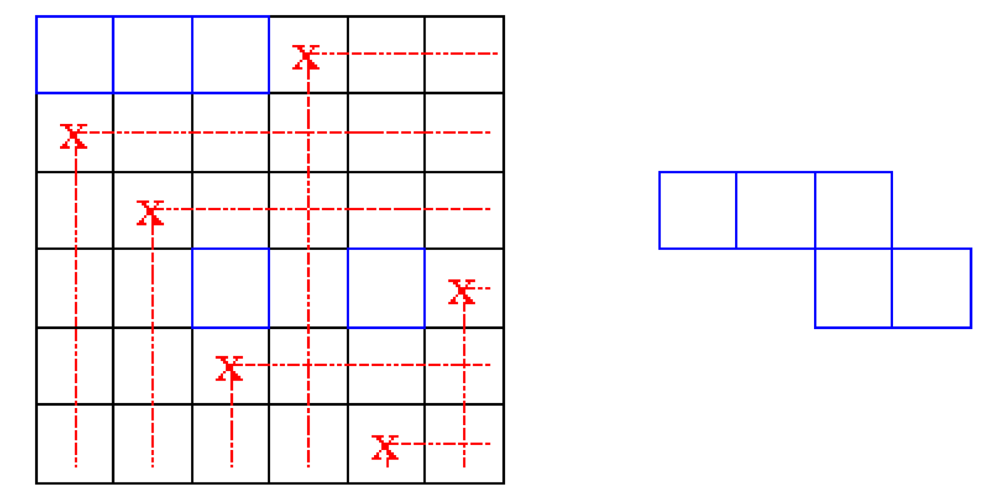
$$R(25134) = \{\sigma_4\sigma_3\sigma_1\sigma_2, \sigma_4\sigma_1\sigma_3\sigma_2, \sigma_1\sigma_4\sigma_3\sigma_2\}$$
$$D(R(25134)) = \{(1,2), (1,3), (2,3)\}$$

Since $D(R(4132)) = D(R(25134))$, we have $F_{4132} = F_{25134}$.

Note: If $F_{w_1} = F_{w_2}$, then it is not necessarily the case that $w_1, w_2 \in S_n$. It is the case that if $F_{w_1} = F_{w_2}$, then $l(w_1) = l(w_2)$.

Permutation Diagrams

Every permutation $w \in S_n$ has a corresponding diagram, denoted $D(w)$. The construction of $D(412635)$ is shown below:



If $w \in S_n$ is 321-avoiding, then $D(w)$ is a skew shape.

Known Equalities

Given $w_1 = (x_1, \dots, x_n) \in S_n$ and $w_2 = (y_1, \dots, y_m) \in S_m$, we define $w_1 \to w_2$ as

$$w_1 \to w_2 := (x_1, \dots, x_n, (y_1 + n), \dots, (y_m + n)) \in S_{n+m}.$$

Theorem: $F_{w_1 \to w_2} = F_{w_1} F_{w_2}$. This corresponds to combining the diagrams $D(w_1)$ and $D(w_2)$ at their tips.

$$F_{312 \to 3142} = F_{312} F_{3142} = F_{3142 \to 312}$$

Given $w \in S_n$, we define $w^* := w_0 w^{-1} w_0$, where w_0 is the element of maximal length in S_n.

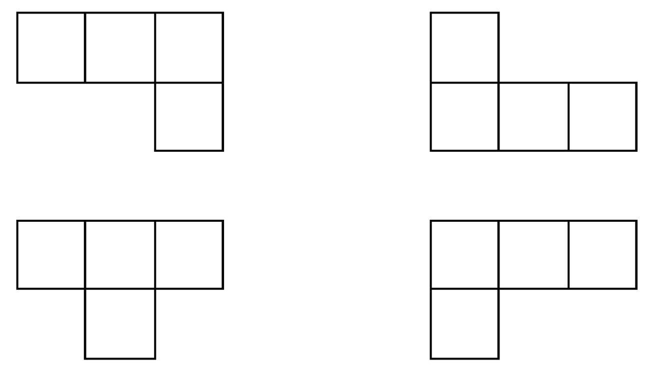
Theorem: For all $w \in S_n$, we have $F_w = F_{w^*}$. If w is 321-avoiding, this corresponds to rotating the diagram $D(w)$ 180°.

$$F_{41253} = F_{41253^*} = F_{25134}$$

Conjectures

We have that the following Stanley symmetric functions are all equivalent:

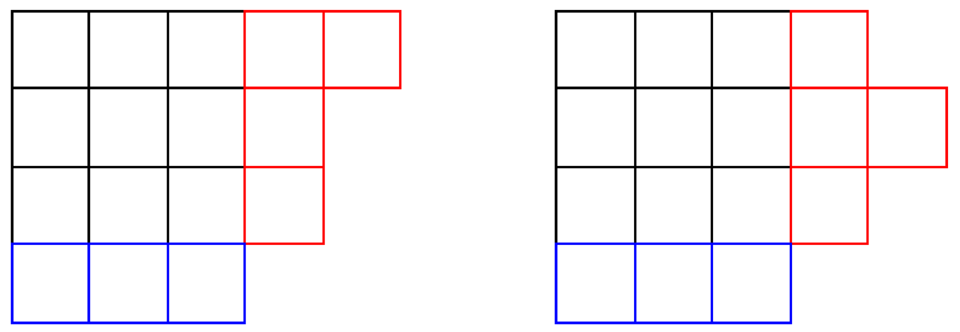
$$F_{41253} = F_{25134} = F_{4132} = F_{4213}$$



Conjecture: If $D(w_1)$ and $D(w_2)$ are equivalent under row and column permutations, then $F_{w_1} = F_{w_2}$. We know the converse is false.

We are also interested in cases where $F_{w_1 \leftarrow w_2} = F_{w_3 \leftarrow w_4}$.
Conjecture: If $D(w_1)$ and $D(w_2)$ are equivalent under row and column permutations and $w_1, w_2 \in S_n$, then $F_{w_1 \leftarrow w} = F_{w_2 \leftarrow w}$.

$$F_{3241 \leftarrow 4123} = F_{2431 \leftarrow 4123}$$



Question: How are Stanley symmetric functions related to skew Schur functions? In particular, for a given $w \in S_n$, we want to know if $F_w = F_{w'}$ for some w' such that $D(w')$ is a skew shape.

References

- Richard P. Stanley. On the number of reduced decompositions of elements of Coxeter groups. *European J. Combin.*, 5(4):359-372, 1984.
- Sergey Fomin, Curtis Greene, Victor Reiner, and Mark Shimozono. Balanced labellings and Schubert polynomials. *European J. Combin.*, 18(4):373-389, 1997.