Fall 2009 Research Report

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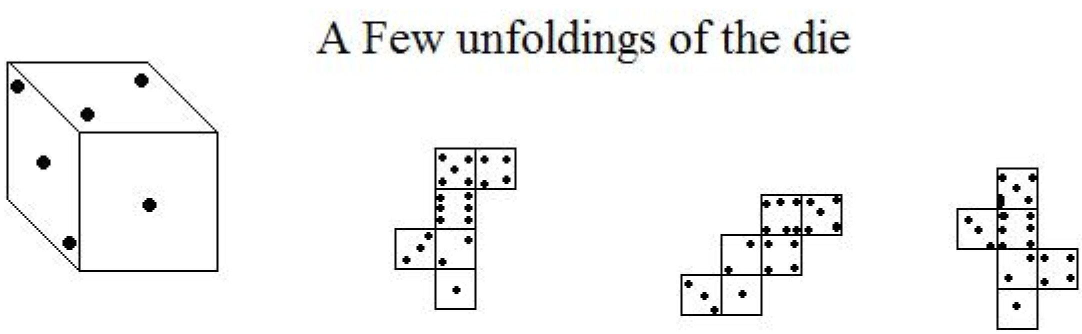
Introduction

In order to understand my research, one must first understand the definition of an unfolding of a given polyhedron.

Def: An unfolding of a given polyhedron is a two-dimensional representation of that polyhedron created by cutting a number of edges of the polyhedron and flattening it into the 2-D plane so that a) the polyhedron is not cut into two or more pieces and b) no corner is allowed to stick out of the plane.

As is often the case with visualization, an example will be helpful:



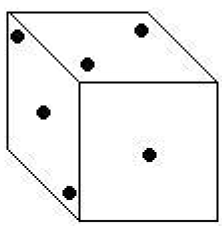
The ultimate goal of my research was to calculate the total number of possible unfolding of various polyhedra (and classes of polyhedra; for instance, the right pyramids with n-sided bases). With guidance from Professor McNamara, I utilized two very useful techniques – Kirchoff's Spanning Tree Theorem and Burnside's Lemma – in order to calculate these numbers, some of which seem to have been previously unknown (or little known).

Choices

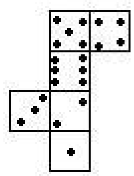
When calculating the number of unfolding of a given polyhedron, there are two choices to be made, each of which affects the final result.

Choice 1: Do we treat congruent faces as distinguishable or indistinguishable?

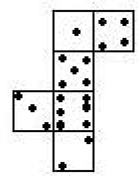
Consider our die, and the first unfolding of it (#1). It is possible to unfold the die as shown by #2:



#1



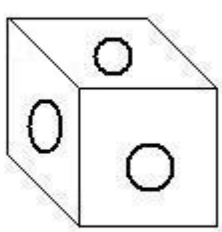
#2



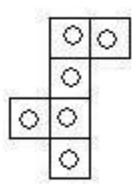
The shapes of #1 and #2 are the same, but the faces are in a different order. We need to decide whether or not to count unfolding #1 and unfolding #2 as the same.

Choice 2: Do we treat the inside and outside of the polyhedron as distinguishable or indistinguishable?

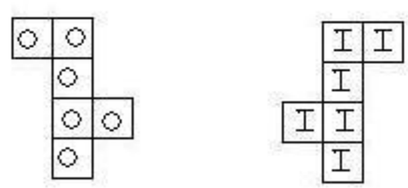
Once again, consider our die. Ignoring the pips for a moment, suppose each face has an O printed on the outside. Furthermore, let the die be hollow, and suppose the inside of each face has an I printed on it. In this setup, unfolding #1 is as below. We can unfold our die as shown in #3; if we flip this over, we get the same shape as #1, but with the inside up rather than the outside.



#1



#3



Again, we need to make a decision about whether or not unfolding #1 is the same as unfolding #3.

Depending on our two choices, we have four possible cases, which for the rest of the paper will be referred to as follows:

Case 1: Faces distinguishable, inside/outside distinguishable.

Case 2: Faces indistinguishable, inside/outside distinguishable.

Case 3: Faces indistinguishable, inside/outside indistinguishable.

Case 4: Faces distinguishable, inside/outside indistinguishable.

Beginnings

The first method we used to try to count the number of unfolding of a given polyhedron was to simply list them as they came to mind. This proved to be quite difficult for two reasons. First of all, there didn't seem to be any way to determine when the list was complete; second, depending on what case we were dealing with, it was not always obvious which seemingly different unfoldings were the same. We were able to find some numbers, but our confidence in them was tenuous at best. Professor McNamara found two tools, however, that would allow us to count unfoldings with a high degree of certainty for a given polyhedron and a given case (out of the 4 previously described).

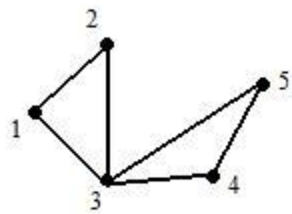
Methods

Case 1 (Matrix-Tree Theorem)

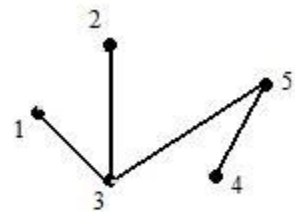
Case 1 is the easiest to deal with, because we need not worry about any two unfoldings being counted as the same; in order to understand the method in case 1, however, we need something called the Matrix-Tree Theorem. This theorem requires a basic understanding of some terms from graph theory.

Def: A <u>graph</u> is a (labeled) set of vertices and edges connecting those vertices.

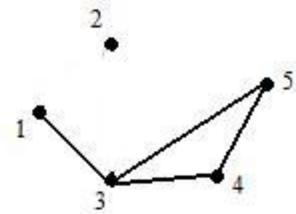
Def: A <u>spanning tree</u> of a given graph is a connected subset of the edges of that graph such that every vertex is hit by at least one edge, and there are no cycles (closed loops) contained in the subset.



A graph, G

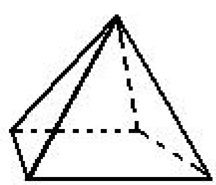


A Spanning Tree of G

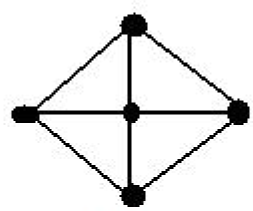


Not a Spanning Tree of G

There is a well known fact that we must now utilize: it is possible to represent any given polyhedron by a graph. For example, consider the right pyramid with a square base. We can represent this as the graph below (labeled pyr-4).



Square Pyramid



pyr-4

Notice the graph has the same number of vertices and edges as the pyramid, and the same number of faces, if we treat the unenclosed portion of the plane as the square base.

The next observation we make is more subtle but immediately believable: every case 1 unfolding of a polyhedron corresponds uniquely to a spanning tree of the associated graph. This follows from our characterization of unfolding as a cutting of edges; if we take a subset of the edges of the graph, cutting along the corresponding edges of a polyhedron will produce an unfolding if and only if the subset was a spanning tree! If the first condition of a spanning tree was not satisfied, we would retain a 3-D corner that could not be flattened into the 2-D plane. If the second condition is not satisfied, and our set of edges contains a cycle, then we will have cut our polyhedron into two separate pieces.

So, as far as case 1 goes, we have transformed the problem: to count the unfoldings of a polyhedron, we need only count the distinct spanning trees of the associated graph. It turns out that Kirchoff's Spanning Tree Theorem tells us exactly how to do this.

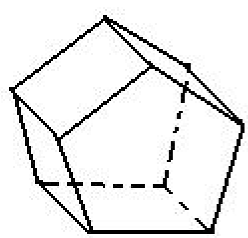
KIRCHOFF'S SPANNING TREE THEOREM

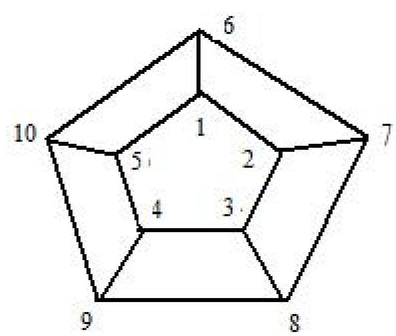
Suppose we have a graph consisting of vertices v_1, v_2, … v_n and edges e_1, e_2, … e_m. We now create an nxn matrix, L, as follows:

If i = j, then L_{ij} is the number of edges touching v_i. If i does not equal j and if there is an edge connecting v_i and v_j, both L_{ij} and L_{ji} are -1; if there is no such edge, both L_{ij} and L_{ji} are 0.

Now, let L' be the n-1xn-1 matrix formed by omitting the last row and column of L (this is for uniformity in the rest of the paper; the following is true if you omit the m^{th} row and m^{th} column of L). The determinant of this matrix is equal to the number of spanning trees of our graph.

For a full example of case 1, let us find the number of unfoldings of a right pentagonal prism, the graph of which is shown below.





The labeling of the vertices from 1 to 10 is arbitrary. We construct L now, and then omit the last row/column to get L':

The Matrix L'

$$\begin{bmatrix} 3 & -1 & 0 & 0 & -1 & -1 & 0 & 0 & 0 \\ -1 & 3 & -1 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & -1 & 3 & -1 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & -1 & 3 & -1 & 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & -1 & 3 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 3 & -1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & -1 & 3 & -1 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & -1 & 3 & -1 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & -1 & 3 \end{bmatrix}$$

The determinant of L’ is 1,805; thus, there are 1,805 ways to unfold a pentagonal prism in case 1 (refer to table at end of paper).

Case 2 (Burnside’s Lemma)

Recall that in case 2, we now treat congruent faces as indistinguishable, though the inside & outside are still distinguishable. This case is harder than the previous one, because now some different spanning trees may correspond to the same unfolding. It turns out that two spanning trees give the same case 2 unfolding precisely when the second spanning tree can be attained from rotating the first.

These two spanning trees of the right triangular prism define the same unfolding

In this case, the first thing we need to do is look at the group of rotations of a given polyhedron. Once we’ve done this, we can use a very useful theorem: Burnside’s Lemma.

BURNSIDE’S LEMMA (in the context of case 2).

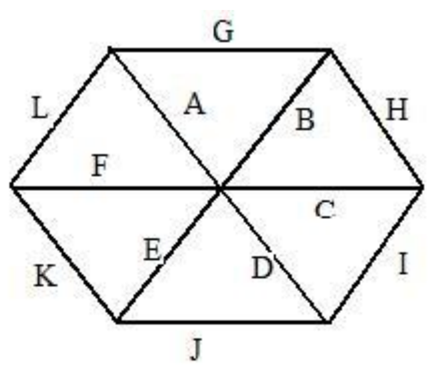
Let R_0, R_1, … R_{n-1} be the rotations on a given polyhedron. Then the number of case 2 unfoldings of the polyhedron equals $\frac{1}{n}(T_0 + T_1 + \cdots + T_{n-1})$, where T_i is the total number of spanning trees that remain unchanged by R_i.

Example (Hexagonal Pyramid)

Consider the hexagonal pyramid and associated graph.

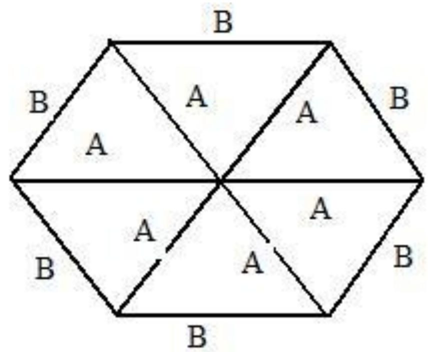
Right Hexagonal Pyramid
and Graph

There are 6 rotations we can apply to this pyramid that result in the same orientation of the polyhedron: these are the 60°, 120°, 180°, 240°, 300°, and identity (0°) rotations about the axis going through the point of the pyramid and the center of the base. We would like to count the number of spanning trees that are fixed by (that is, remain the same after) each rotation. Since the identity rotation does nothing, the number of spanning trees of the graph fixed under this rotation is simply the total number of spanning trees of the graph (which we can easily compute using Kirchoff’s Spanning Tree Theorem). For each of the other rotations, we’ll first label the *edges* of the graph:



We now attack each rotation individually. For the 60^{o} rotation, a spanning tree that includes A will only be fixed if it includes B, C, D, E, and F as well. Similarly, if a spanning tree contains G, it is fixed only if it contains H, I, J, K, and L. since we have groups of edges that must be selected together, we can relabel the edges as follows:

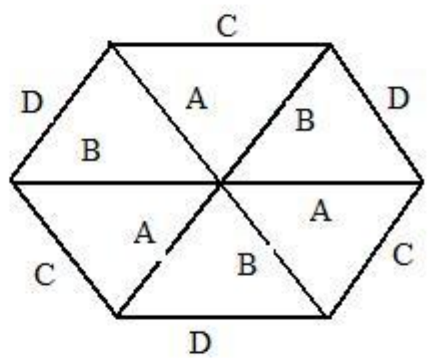
60 degree rotation



If we choose the sides labeled A, we have a spanning tree. If we choose the sides labeled B, we have a cycle, which is not allowed. So, the 60^{o} rotation fixes a single spanning tree.

Next, we'll look at the 120^{o} rotation. By the same reasoning as before, it yields the following relabeling:

120 degree rotation



If we select A and B, A and C, A and D, B and C, or B and D, we get a spanning tree (C and D contains a cycle). So, the 120^{o} rotation fixes five spanning trees. Proceeding in this manner, we get the following number of spanning trees fixed by the rotations:

Rotation	0^{o}	60^{o}	120^{o}	180^{o}	240^{o}	300^{o}
Trees Fixed	320	1	5	16	5	1

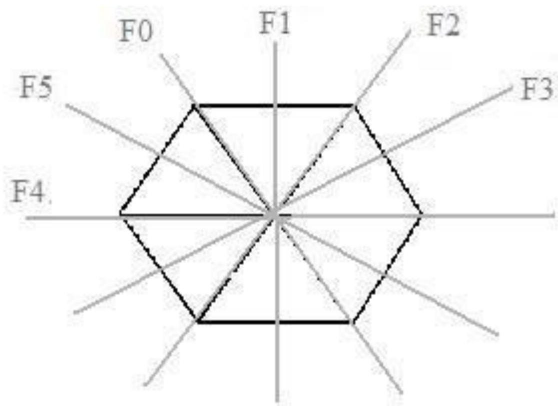
So, the total number of case 2 unfoldings of a right hexagonal pyramid is

$$\frac{1}{6}(320 + 1 + 5 + 16 + 5 + 1) = 58.$$

Case 3

Determining the number of case 3 unfoldings of a given polyhedron is more difficult still. The technique is the same as in case 2, but now we need to consider the group of "flips" (reflections) as well as the group of rotations.

Axes and Flips of the Hexagonal Pyramid

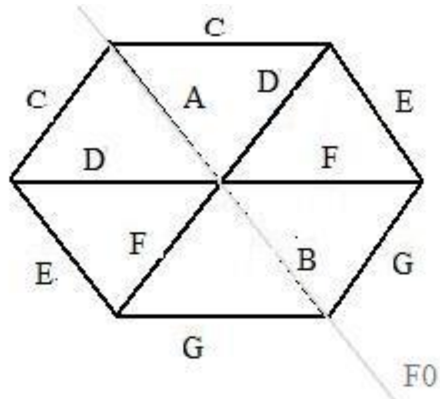


For case 3, we rephrase Burnside's Lemma Slightly:

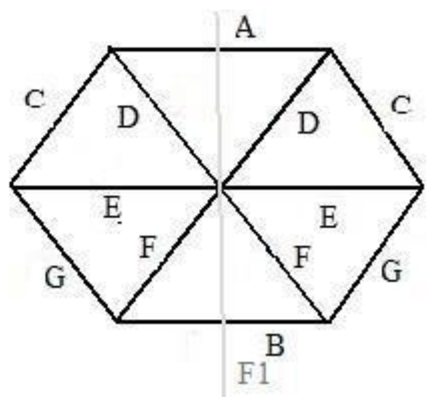
BURNSIDE'S LEMMA (in the context of case 3)

Let R_0, R_1, … R_{n-1} be the rotations on a given polyhedron, and let F_0, F_1, … F_{n-1} be the group of flips on the polyhedron. Then the number of case 2 unfoldings of the polyhedron equals $\frac{1}{2n}(T_0 + T_1 + \cdots + T_{n-1} + T_0^* + T_1^* + \cdots + T_{n-1}^*)$, where T_i is the total number of spanning trees that remain unchanged by R_i and T_i* is the total number of spanning trees that remain unchanged by F_i.

We will return to the example of the right hexagonal pyramid, as we have already calculated all of the values of T, and need only calculate the T*s. Analogously to case 2, we look at each flip individually. We'll now look at F_0; as before, we have assigned the same letter to all edges in a group that must be chosen together. Noting that, by definition, a spanning tree of a graph with n vertices contains n-1 edges, we see that we must either select both A and B or neither A and B (because we need 6 edges, an even number). If we do not select A and B, it is impossible to select a spanning tree from among the remaining edges. So, having selected A and B, we must now select two of C, D, E, F, and G. The only pairs that do not yield spanning trees (because they contain cycles) are CD and GF; this means that F_0 fixes the 8 other spanning trees attainable. It is easy to see that, because of symmetry, F_2 and F_4 will fix the same number.



We next look at F_1 (because of symmetry, this will fix the same number of spanning trees as F_3 and F_5) . Again, we have relabeled the graph. As before, we see that we must either select both A and B or neither A nor B. This time, however, selecting both A and B will necessarily give us a cycle, so we must omit both. Selecting from the remaining sets of edges, it turns out there are 8 that will give us a fixed spanning tree (only CDE and EFG fail).



So, we now have the following table of values:

Rotation	0^o	60^o	120^o	180^o	240^o	300^o
Trees Fixed	320	1	5	16	5	1
Flip	F_0	F_1	F_2	F_3	F_4	F_5
Trees Fixed	8	8	8	8	8	8

This tells us that the total number of case 3 unfoldings of a right hexagonal pyramid is

$$\frac{1}{12}\big(320+1+5+16+5+1+6(8)\big) = 33.$$

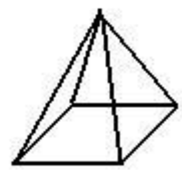
A Note on Case 4

The research I conducted this semester did not address case 4 (where congruent faces are distinguishable, but the inside and outside of the polyhedron are indistinguishable). This case seems to be of the same difficulty as case 2.

Going Forward

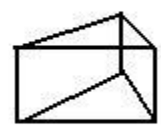
This research leaves a lot of questions to be answered, and, indeed, asked. Our research focused on calculating numbers, but one topic to address in the future would be the relations between these sets of numbers. Is it possible or easy to asymptotically bound the number of unfoldings of a given class of polyhedra, so that one could attain a good estimate of the number of unfoldings without having to do calculations for very large n? Furthermore, some of the sequences of numbers we discovered are known, but for reasons (seemingly) completely unrelated to unfoldings of polyhedra, or, indeed, to any sort of dealings with polyhedra. Why would those numbers turn up in this research?

Appendix of values



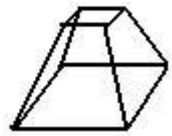
Right Pyramid with n-sided base

n	Case 1	Case 2	Case 3
3	16	6	4
4	45	13	8
5	121	25	15
6	320	58	33
7	841	121	67
8	2205	283	152
9	5776	646	320
10	15125	1527	791



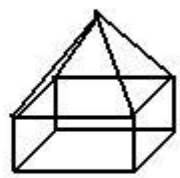
Right n-prism (same as Bipyramid with n-sided generator)

N	Case 1	Case 2	Case 3
3	75	15	9
4	384	52	29
5	1805	190	?
6	8100	690	?
7	35287	2556	?
8	150528	9464	?
9	632025	35245	?
10	2620860	131253	?



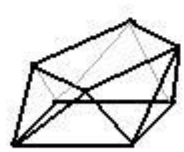
Truncated n-pyramid

N	Case 1	Case 2	Case 3
3	75	25	14
4	384	96	54
5	1805	361	186
6	8100	1350	690
7	35287	5041	2541
8	150528	18816	?
9	632025	70225	?
10	2620860	262086	?



Extended n-prism

N	Case 1	Case 2	Case 3
3	361	121	63
4	3509	885	448
5	30976	6196	?
6	261725	43691	?
7	2163841	309121	?
8	17688869	2211555	?
9	143736121	15970761	?
10	1164201984	116423308	?



n-sided Anti-prism

N	Case 1	Case 2	Case 3
3	384	72	?
4	3528	462	?
5	30250	3080	?
6	248832	?	?
7	1989806	?	?
8	15586704	?	?
9	120187008	?	?
10	915304500	?	?