RIGIDITY OF RIGHT-ANGLED COXETER GROUPS

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Abstract

In mathematics, a *group* is the set of symmetries of an object. Coxeter groups are a broad and natural class of groups that are related to reflectional symmetries. Each Coxeter group is determined by a diagram, called a *labeled graph*, that encodes algebraic information about the group. In general, two different labeled graphs can give rise to the same group. It is therefore natural to ask: *are there classes of Coxeter groups that have unique associated graphs?* Coxeter groups that have a unique labeled graph are said to be *rigid*. There are important classes of Coxeter groups that are rigid. Radcliffe [5] showed that the class of *right-angled* Coxeter groups is rigid, and Bahls [1] extended this result to the class of *even* Coxeter groups. The main aim of this thesis is to provide an alternative proof, based on an argument outlined by A. Piggott [4], of the rigidity of right-angled Coxeter groups.

1. Introduction

In mathematics, a *group* is the set of symmetries of an object. Coxeter groups are a broad and natural class of groups that are related to reflectional symmetries. Each Coxeter group is determined by a diagram, called a *labeled graph*, that encodes algebraic information about the group. We will see that in general, two different labeled graphs can give rise to the same group. It is therefore natural to ask: *are there classes of Coxeter groups that have unique associated graphs?* Coxeter groups that have a unique labeled graph are said to be *rigid*. There are important classes of Coxeter groups that are rigid. Radcliffe [5] showed that the class of *right-angled* Coxeter groups is rigid, and Bahls [1] extended this result to the class of *even* Coxeter groups. The main aim of this thesis is to provide an alternative proof, based on an argument outlined by A. Piggott [4], of the rigidity of right-angled Coxeter groups.

A right-angled Coxeter group $W(\Gamma)$, is determined by a simple (unlabeled) graph Γ: we often write W instead of $W(\Gamma)$ when it is clear from the context (the precise construction is given in Section 2.4). Our main theorem is the following.

Theorem 1.1 (Rigidity of right-angled Coxeter groups)**.** *Let Γ_1 and Γ_2 be simple graphs. Then the right-angled Coxeter groups $W(\Gamma_1)$ and $W(\Gamma_2)$ are isomorphic groups if and only if Γ_1 and Γ_2 are isomorphic graphs.*

The thesis is organized as follows. In Section 2 we motivate a discussion of groups with an example; we also introduce the notion of rigidity. In Section 3 we collect useful properties of right-angled Coxeter groups. In Section 4 we construct two graphs from a pair (Γ, W), where $W = W(\Gamma)$; a graph Ω from the graph Γ and a graph Δ from the group W. This is illustrated in the following diagram, where $\rightarrow$ indicates a deterministic procedure:

$$\begin{array}{ccc} \Gamma & \rightarrow & W \\ \downarrow & & \downarrow \\ \Omega & & \Delta \end{array}$$

In Section 4 we show that the graphs Ω and Δ are isomorphic. This shows that W determines Ω and improves our diagram to the following:

$$\begin{array}{ccc} \Gamma & \rightarrow & W \\ \downarrow & & \downarrow \\ \Omega & \cong & \Delta \end{array}$$

In Section 5 we reconstruct the graph Γ using only Ω. Our diagram becomes:

$$\begin{array}{ccc} \Gamma & \rightarrow & W \\ \updownarrow & & \downarrow \\ \Omega & \cong & \Delta \end{array}$$

So given a right-angled Coxeter group $W = W(\Gamma)$, there is a unique associated graph Ω. In Section 5.3 we show that the construction of Ω from the graph Γ is invertible. It follows that for each right-angled Coxeter group there is only one graph Γ, associated to the group. Equivalently, non-isomorphic graphs give rise to non-isomorphic right-angled Coxeter groups, and the theorem is proved.

Section 6 gives a worked example which we encourage the reader to consult at appropriate stages of the proof.

Mathematicians often look for multiple proofs of the same theorem. A proof is not just a means to establish veracity; it provides insight into the inner workings of a problem. This proof is novel because, for the most part, it relies on elementary graph theory. This is useful because people tend to have a good intuition for pictures and diagrams. As such our proof is accessible to a broad mathematical audience.

2. Preliminaries

In this section we use an example to motivate a discussion of *groups*, *group presentations* and *words*. We also formally introduce *graphs*, *right-angled Coxeter groups* and the question of rigidity. Our example will illustrate that, in general, Coxeter groups are not rigid.

2.1. **Symmetry and Groups.** There are various ways that the coordinate plane can be mapped onto itself while preserving the distance between points. For example, it can be translated, rotated, and reflected. Such distance preserving mappings are called *isometries* of the plane. Consider a hexagon in the plane as shown below.

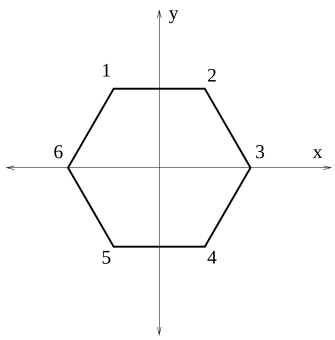

Figure 1. Hexagon.

The isometries of the plane that leave the hexagon unchanged to our perception are called *symmetries* of the hexagon. In this case the symmetries include certain rotations, reflections, and the trivial act of mapping each point of the hexagon back to itself. For example, if the plane is flipped over the y-axis the hexagon and plane will remain visually unchanged. However, if the corners of the hexagon were labeled we would know that the hexagon had been changed. The symmetries of the hexagon include reflections over the the lines through opposite corners and over the lines through the midpoints of opposite sides. It can also be rotated about its center by 60, 120, 180, 240, 300 or 360 degrees. The location of two adjacent corners will determine the position of the hexagon. Since a corner has six possible locations and an adjacent corner can either be clockwise adjacent or counterclockwise adjacent, we know that there are twelve symmetries in total. Let's denote the set of the twelve symmetries of the hexagon by H.

Consider rotating the plane clockwise by $60°$ and then reflecting it over the y-axis. When each of these symmetries is done one after the other the result is still a symmetry. This illustrates a natural way to define the *product* of two symmetries: given two symmetries α and β, the product $\alpha * \beta$ is defined as performing β then α. This operation is a *binary operation* on the set H. A binary operation on a set S is a function $f : S \times S \to S$. Addition and multiplication are both examples of binary operations. If the plane is rotated clockwise by $60°$ and then reflected over the y-axis, corner 1 will be sent to corner 2 then back to itself. However, if the plane is reflected over the y-axis and then rotated clockwise by $60°$, corner 1 will be sent to corner 2 and then to corner 3's original position. This shows that, unlike addition where $a + b = b + a$, the operation $*$ is not commutative; that is $\alpha * \beta$ is not necessarily equal to $\beta * \alpha$.

Let's observe some other properties of the set H and operation $*$.

(1) **Identity**: Rotating the plane by 0 or $360°$ is called the trivial symmetry, and denoted is 1. If we perform symmetry α then 1, it's the same as 1 then α, which is the same as just performing α. That is to say $\alpha * 1 = 1 * \alpha = \alpha$.

(2) **Associativity**: Note how we defined $*$. For $\alpha * \beta$ we first do β then α. Consider $(\alpha * \beta) * \gamma$. This means perform γ then $(\alpha * \beta)$. It amounts to performing γ then β then α. Now consider $\alpha * (\beta * \gamma)$. This would amount to doing γ then β then α as well. Thus, regardless of the parenthetical grouping, $(\alpha * \beta) * \gamma = \alpha * (\beta * \gamma)$. This property is known as *associativity*.

(3) **Inverse**: When we perform symmetry α, each point of the hexagon is mapped to a distinct point in the plane. This mapping can be undone by mapping each point back to its original position. Since α is a symmetry, the undoing of α is a symmetry as well. The undoing of α is called α *inverse*, and written α^{-1}. Observe that doing α then α^{-1} is the identity 1.

These properties might be familiar. The integers have these properties under the operation of addition with identity element 0. In fact it is these three properties that precisely define what is called a *group*.

Definition 2.1. *A* **group** *is a set H with a binary operation $*: H \times H \to H$ such that there is an identity element, $*$ is associative, and each element in H has an inverse.*

From now on we will refer to symmetries as being elements of a group, and will write $\alpha\beta$ instead of $\alpha * \beta$. The identity element of a group will be denoted by 1. The inverse of the element α will be denoted by α^{-1}.

2.2. **Generators, Relations and Words.** The group H includes an element ρ which is a rotation by $60°$. Consider ρ^2. The product of ρ with itself is a $120°$ rotation. Repeatedly taking the product of ρ with itself generates the five other rotations including the identity. Below is another figure of the hexagon with two lines of reflection. We call the reflection over the line a, α, and the reflection over the line b, β.

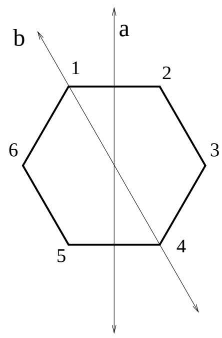

FIGURE 2. Hexagon with Lines of Reflection.

Let's compute the product $\alpha\beta$. Reflecting over b will send corner 1 to itself, then reflecting over a will send 1 to 2's original position. Corner 2 will be sent to 6 by β then to 3's original position by α. From this we can tell that $\alpha\beta = \rho$, where ρ is a $60°$ rotation. By repeatedly taking the product of α and β we can get all the rotations in H. Similar calculations show that any reflection in H can be achieved using only α and β. We say that α and β are *generators* of the group.

The product $\alpha\alpha$ is 1, since doing a reflection twice amounts to the identity. Similarly $\beta\beta = 1$. Elements that are their own inverse are called *involutions.*

Since $\alpha\beta$ is a 60° rotation, $(\alpha\beta)^6 = 1$. Equations involving group elements are *relations.* It is perhaps not obvious but specifying the generators α, β and the relations $\alpha^2 = 1, \beta^2 = 1$ and $(\alpha\beta)^6 = 1$ uniquely determines the group H; that is, all multiplication rules in the group can be deduced from this list of relations. We denote this group succinctly as follows:

$$\langle \alpha, \beta | \alpha^2 = 1, \beta^2 = 1, (\alpha\beta)^6 = 1 \rangle. \tag{1}$$

This is known as a *group presentation.* In general, it's hard to find generators and relations that determine a given group. Groups can be described in terms of generators and relations in multiple ways. Determining if two different sets of generators and relations give rise to the same group is usually a very difficult problem.

In a finitely generated group every group element can be written as the product of generators and their inverses. When the generators are involutions, as in the presentations considered in this thesis, each element can be written as a product of generators. We can think of generators as *letters*, and finite strings of generators as *words* which *spell* the group element corresponding to that product of generators. We consider words to be *equivalent* if they spell the same group element. We use relations to determine which words are equivalent. The set of words that spell a certain group element is called an equivalence class of words, so we can think of group elements as equivalence classes of words. The empty string represents the identity element in the group.

Consider the presentation of H in equation (1). In this case, our letters are α and β. Any finite string of α's and β's is a word which determines a group element. Some distinct words determine the same group element, and so are equivalent. For example, the words: $\beta\beta\alpha\beta\alpha\beta\alpha\beta$ and $\alpha\alpha\beta\alpha\beta\alpha$ are equivalent. We see this as follows. Since α and β are involutions we know

$$\beta\beta\alpha\beta\alpha\beta\alpha\beta\alpha\beta = \alpha\beta\alpha\beta\alpha\beta\alpha\beta \text{ and } \alpha\alpha\beta\alpha\beta\alpha = \beta\alpha\beta\alpha.$$

Since $(\alpha\beta)^6 = 1$ we know that:

$$\alpha\beta\alpha\beta\alpha\beta\alpha\beta(\alpha\beta\alpha\beta) = 1.$$

If we multiply on the right by $\beta\alpha\beta\alpha$ and use the fact that the letters are involutions we get:

$$\alpha\beta\alpha\beta\alpha\beta\alpha\beta = \beta\alpha\beta\alpha.$$

Thus we can conclude that

$$\beta\beta\alpha\beta\alpha\beta\alpha\beta\alpha\beta = \alpha\alpha\beta\alpha\beta\alpha.$$

We define the *length of a word* to be the number of letters in the word. So $\alpha\alpha\beta\alpha\beta\alpha$ has length 6 and $\beta\alpha\beta\alpha$ has length 4, even though they both spell the same group element. We define the *length of a group element* to be the minimum number of letters needed to spell that element. Let w be the group element spelled by $\alpha\alpha\beta\alpha\beta\alpha$. Since w is spelled by $\beta\alpha\beta\alpha$, and no shorter word can spell w, the length of w is 4. We write $\ell(\alpha\alpha\beta\alpha\beta\alpha) = 6$ and $\ell(w) = 4$. We say that a word is *reduced* if there is no shorter way to spell the same group element. For example, $\beta\alpha\beta\alpha$ is reduced and $\alpha\alpha\beta\alpha\beta\alpha$ is not.

The group H was easily described without generators or relations. In general, however, groups can be very large or even infinite. It is difficult to list all the elements in large groups and impossible to list all the elements and products of an infinite group. Group presentations and words are a useful way to think about and describe such unwieldy groups.

2.3. **Graphs.** Here we define simple graphs and several graph properties that are important to this thesis.

Definition 2.2. *A* **(simple) graph** Γ*, consists of a pair of sets* (V_Γ, E_Γ)*. The set* $V_\Gamma = \{v_1, \ldots, v_n\}$ *is a set of* **vertices***. The set* E_Γ*, the set of* **edges***, is a set of* *unordered pairs from V_Γ; that is, $E_\Gamma \subseteq \{\{v_i, v_j\} \mid v_i, v_j \in V_\Gamma$ and $v_i \neq v_j\}$. We say v_i and v_j are* **adjacent** *if $\{v_i, v_j\} \in E_\Gamma$.*

For an example of a graph see Figure 5.

Definition 2.3. *Let $\Gamma = (V_\Gamma, E_\Gamma)$ be a graph. A* **subgraph** *of Γ is a graph $\Delta = (V_\Delta, E_\Delta)$ where $V_\Delta \subseteq V$ and $\{v_i, v_j\} \in E_\Delta$ if and only if $\{v_i, v_j\} \in E$.*

Definition 2.4. *A graph $\Gamma = (V_\Gamma, E)$ is* **complete** *if whenever $v_i, v_j \in V$ and $v_i \neq v_j$, $\{v_i, v_j\} \in E$ (all the vertices are pairwise adjacent).*

When graphs are essentially the same we say they are isomorphic. We make the following more precise definition.

Definition 2.5. *Let $\Gamma = (V_\Gamma, E_\Gamma)$ and $\Psi = (V_\Psi, E_\Psi)$ be graphs. A* **graph isomorphism** *from Γ to Ψ is a bijective map $f : V_\Gamma \to V_\Psi$ such that $\{v_i, v_j\} \in E_\Gamma$ if and only if $\{f(v_i), f(v_j)\} \in E_\Psi$. If such a graph isomorphism exists, then we say that Γ and Ψ are* **isomorphic***.*

2.4. **Right-Angled Coxeter Groups and The Rigidity Question.** A right-angled Coxeter group is a group determined by a simple graph. The graph encodes all the information contained in a group presentation. Each vertex corresponds to a generating involution and the edges indicate which generators commute.

Definition 2.6. *Let $\Gamma = (V_\Gamma, E_\Gamma)$ be a graph. The* **right-angled Coxeter group** *associated to Γ is the finitely generated group $W(\Gamma)$, with generators V_Γ and relations $v_i^2 = 1$ for each i and $(v_i v_j)^2 = 1$ if and only if $\{v_i, v_j\} \in E_\Gamma$. That is to say, all the vertices correspond to generators that are involutions and two generators commute if and only if they are adjacent.*

Recall that we often write W instead of $W(\Gamma)$ when Γ is clear from the context. A right-angled Coxeter group is *rigid* if there is a unique associated graph.

2.5. **A Non-Rigid Coxeter Group.** Right-angled Coxeter groups are a special case of a broader class of groups called Coxeter groups. A Coxeter group is determined by a *labeled graph*; a graph where each edge is labeled with an integer greater than or equal to two. Recall the group H from section 2.2 with presentation $\langle \alpha, \beta | \alpha^2 = 1, \beta^2 = 1, (\alpha\beta)^6 = 1 \rangle$. It is a Coxeter group but not a right-angled Coxeter group. Its presentation is encoded graphically as two points which represent α and β, with an edge between them labeled 6 to express that $(\alpha\beta)^6 = 1$ (see Figure 3).

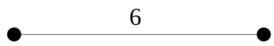

FIGURE 3. Graphical Presentation of H.

Recall the hexagon in Figure 2. Let γ be a rotation by 180° and σ the reflection over the line through the midpoints of segments 61 and 34. Then γ, σ and α also generate H. These generators provide an alternate presentation of H:

$$\langle \alpha, \sigma, \gamma | \alpha^2 = 1, \sigma^2 = 1, \gamma^2 = 1, (\alpha\gamma)^2 = 1, (\sigma\gamma)^2 = 1, (\alpha\sigma)^3 = 1 \rangle.$$

The graph associated with this presentation is a triangle with two sides labeled 2 and one side labeled 3.

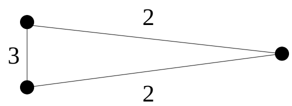

FIGURE 4. Another Graphical Presentation of H.

This example shows that, in general, Coxeter groups are not rigid.

3. Properties of Right-Angled Coxeter Groups

It this section we state and prove several properties of right-angled Coxeter groups that will be used in our proof of rigidity.

3.1. **Subgraphs and Subgroups.** Let Γ be a graph and $W(\Gamma)$ the associated right-angled Coxeter group. Each subgraph Δ of Γ determines a subgroup of $W(\Gamma)$ generated by V_Δ. The following theorem asserts that all finite subgroups are conjugate to subgroups generated by complete subgraphs.

Theorem 3.1 ([2](Chapter 4, Exercise 2(d)))**.** *If G is a finite subgroup of a right-angled Coxeter group $W(\Gamma)$, then there exists $w \in W(\Gamma)$ and a complete subgraph $X \subseteq \Gamma$, such that $wGw^{-1} = \{wgw^{-1} : g \in G\} = W(X)$.*

3.2. **The Deletion Condition.** The following condition describes when and how we can reduce words corresponding to elements in a right-angled Coxeter group. A proof may be found in [3](Page 10, Theorem 3.1). Recall that a word is reduced if there is no shorter way to spell the corresponding group element.

Lemma 3.2 (Deletion Condition)**.** *If W is a right-angled Coxeter group and $a_1a_2 \dots a_p$ is not a reduced word, then there exist i, j such that $1 \leq i < j \leq p$, $a_i = a_j$ and $a_ia_k = a_ka_i$ for $i < k < j$.*

3.3. **Conjugacy Classes and Cyclically Reduced Involutions.** We now define two properties of group elements.

Definition 3.3. *Given a group element $u \in W$, the* **conjugacy class** *of u, denoted u^W, is the set $\{xux^{-1} : x \in W\}$.*

Observe that if u is a group element with order n and $w \in W$, then wuw^{-1} has order n as well, since

$$(wuw^{-1})^n = wuw^{-1}wuw^{-1} \dots wuw^{-1} = wu^nw^{-1} = ww^{-1} = 1.$$

.

Definition 3.4. *Let $u \in W$ be a group element. We say that u is* **cyclically reduced** *if there is no shorter group element in the conjugacy class of u.*

Note that a *word* can be reduced and a *group element* can be cyclically reduced.

Now we prove two lemmas which establish that cyclically reduced involutions correspond to subsets of the generators that pairwise commute.

Lemma 3.5. *If a reduced word $a_1a_2a_3\dots a_p$ spells a cyclically reduced involution in W, then $a_ia_j = a_ja_i$ for each pair $i, j \in \{1, 2\dots, p\}$.*

Proof. Let $a_1a_2a_3\dots a_p$ be a reduced word that spells a cyclically reduced involution $w \in W$. Since $a_1a_2a_3\dots a_n$ spells an involution, writing it twice spells the identity; that is

$$a_1a_2a_3\dots a_pa_1a_2a_3\dots a_p = 1.$$

For convenience, we relabel this word from 1 to $p+p$ where $a_i = a_{p+i}$,

$$a_1a_2a_3\dots a_pa_{p+1}a_{p+2}a_{p+3}\dots a_{p+p}.$$

Since this word is not reduced, by the deletion condition, there exist i, j with $i < j$, $a_i = a_j$ and $a_ia_k = a_ka_i$ for $i < k < j$.

Suppose that $i < j \leq p$. This implies that

$$a_1a_2\dots\hat{a}_i\dots\hat{a}_j\dots a_pa_{p+1}a_{p+2}\dots a_{p+p} = 1,$$

where the notation $\hat{a}_i$ indicates that a_i has been removed from the word. Since w is an involution if we multiply on the right by $a_1a_2a_3\dots a_p$ we get

$$a_1a_2\dots\hat{a}_i\dots\hat{a}_j\dots a_p = a_1a_2a_3\dots a_p,$$

which contradicts the fact that $a_1a_2a_3\dots a_p$ is reduced.

Similarly, suppose that $p < i < j$. Again this implies that

$$a_1a_2\hat{a}_i\dots\hat{a}_j\dots a_p = a_1a_2a_3\dots a_p$$

which contradicts the fact that $a_1a_2a_3\dots a_p$ is reduced.

Thus we have $i \leq p < j$. Now suppose that $i \neq j - p$; i.e. suppose that a_i is not in the same position in both copies of the word. By the deletion condition, a_i

and a_j commute with a_k whenever $i < k < j$. Since a_i and a_{j-p} are both letters in $a_1a_2a_3 \dots a_p$ and a_{j-p} commutes with a_k whenever $1 \leq k \leq j-p$, we have that:

$$a_1a_2a_3 \dots a_p \text{ is either equivalent to } a_{j-p}a_1 \dots \hat{a}_{j-p} \dots \hat{a}_i \dots a_pa_i$$

$$\text{or equivalent to } a_{j-p}a_1 \dots \hat{a}_i \dots \hat{a}_{j-p} \dots a_pa_i$$

We know $a_{j-p} = a_i$ and that a_i is an involution. So if we conjugate $a_{j-p}a_1 \dots a_pa_i$ by a_i we get $a_1a_2 \dots \hat{a}_i \dots \hat{a}_{p+i} \dots a_p$. This contradicts the assumption that w is cyclically reduced.

Thus we have $i = j - p$; that is or $j = p + i$. Since a_i commutes with a_k for all $k > i$ and $a_j = a_i$ commutes with a_k for all $k < i$, a_i commutes with every letter in $\{a_1, a_2, a_3, \dots, a_p\}$.

We now proceed iteratively. Canceling a_i with a_{p+i} reveals that

$$a_1 \dots \hat{a}_i \dots a_pa_{p+1} \dots \hat{a}_{p+i} \dots a_{p+p} = 1.$$

The deletion condition implies that there exist h, t with $h < t$, $a_h = a_t$ and $a_ha_k = a_ka_h$ for $h < k < t$. Since a_i commutes with every letter we know that

$$a_1 \dots a_i \dots a_h \dots a_pa_{p+1} \dots a_i \dots a_t \dots a_{p+p} = 1.$$

Now similar arguments as before show that $h = t - p$ and that a_t commutes with every letter in $\{a_1, a_2, a_3, \dots, a_p\}$.

□

Lemma 3.6. *If $a_1a_2a_3 \dots a_p$ is a reduced word that spells a cyclically reduced involution $w \in W$, then $a_i \neq a_j$ whenever $1 \leq i < j \leq p$.*

Proof. Suppose $a_1a_2 \dots a_p$ is a reduced word that spells a cyclically reduced involution w. Suppose that $a_i = a_j$ for some i, j, where $1 \leq i < j \leq p$. By our previous Lemma 3.5, the letters in $\{a_1, a_2, a_3, \dots, a_p\}$ pairwise commute. So we know that,

$$a_1a_2 \dots a_i \dots a_j \dots a_p = a_1a_2 \dots \hat{a}_i \dots \hat{a}_j \dots a_pa_ia_j$$

Since $a_ia_j = 1$ we have,

$$a_1a_2 \dots a_ia_j \dots a_p = a_1a_2 \dots \hat{a}_i \dots \hat{a}_j \dots a_p$$

This contradicts the fact that w was reduced. Thus for any i, j with $1 \leq i < j \leq p$ we know that $a_i \neq a_j$. ☐

Note that each generator appears an even number of times in each relation defining W. It follows that each group element has a well-defined notion of parity: if $v_i \in V_\Gamma$ appears an odd (or respectively even) number of times in a word that spells $u \in W$, then v_i appears an odd (or respectively even) number of times in every word that spells u. If w is spelled by $b_1 b_2 \ldots b_q$ then w^{-1} is spelled by $b_q \ldots b_2 b_1$. So letters in w that appear an odd (or respectively) even number of times, appear in w^{-1} an odd (or respectively) even number of times. It follows that letters in a word that spells wvw^{-1}, have the same parity as letters in a word that spells v.

Lemma 3.7. *Each conjugacy class of involutions contains a unique cyclically reduced involution.*

Proof. Recall that every conjugate of an involution is an involution as well. Every conjugacy class contains an element of minimal length and thus contains a cyclically reduced involution. Let u and v be cyclically reduced involutions. Suppose that $u = wvw^{-1}$ for some $w \in W$. Since v and u are cyclically reduced and conjugate they both have the same length. If a letter appears an odd (or respectively even) number of times in a word that spells v then it will appears an odd (or respectively even) number of times in every word that spells u. By Lemma 3.6 every letter in a word that spells a cyclically reduced involution appears exactly once. Thus every letter that appears once in a word that spells u appears once in a word that spells wvw^{-1}. It follows that $w = 1$ and $u = v$. So cyclically reduced involutions that are are conjugate are equal. Thus each conjugacy class of involutions contains a unique cyclically reduced involution. ☐

4. The Graph Ω

In this section we describe two graph constructions.

- Given a graph Γ, construct a graph Ω.
- Given a group W, construct a graph Δ.

If $W = W(\Gamma)$, we then prove $\Omega \cong \Gamma$.

We construct $\Omega = (V_\Omega, E_\Omega)$ as follows:

- V_Ω is the set of complete subgraphs of Γ;
- $\{X, Y\} \in E_\Omega$ if and only if X and Y are distinct complete subgraphs of Γ and X and Y are subgraphs of a complete subgraph Z of Γ.

For an example of a graph Γ and a corresponding graph Ω see Figures 5 and 6.

We construct $\Delta = (V_\Delta, E_\Delta)$ as follows:

- V_Δ is the set of conjugacy classes of involutions in W;
- $\{u^W, v^W\} \in E_\Delta$ if and only if u^W and v^W are distinct conjugacy classes of involutions in W and there exist $x \in u^W$ and $y \in v^W$ such that $xy = yx$.

We represent the various constructions as follows:

$$\begin{array}{ccc} \Gamma & \rightarrow & W(\Gamma) \\ \downarrow & & \downarrow \\ \Omega & & \Delta \end{array}$$

The connection between Ω and Δ is exhibited by the map $i : V_\Omega \to V_\Delta$ defined as follows:

$$i(X) = (\prod_{a_i \in X} a_i)^W. \tag{2}$$

This map sends a complete subgraph X of Γ to the conjugacy class of involutions in W that contains the product of vertices in X. We will show that i induces a graph isomorphism.

Lemma 4.1. *The map $i : V_\Omega \to V_\Delta$ defined in 2, is a bijection.*

Proof. First we show that i is one-to-one. Suppose that $i(X) = i(Y)$. Then $(\prod_{a_i \in X} a_i)^W = (\prod_{b_i \in Y} b_i)^W$. Since all the letters in $\{a_1, a_2, \ldots, a_p\}$ pairwise commute, the word $a_1 a_2 \ldots a_p$ spells an involution in W. Since every letter in $a_1 a_2 \ldots a_p$ appears once and conjugation does not change the parity of letters, there is no shorter word that spells a group element in the same conjugacy class. Thus $a_1 a_2 \ldots a_p$ is a reduced word which spells a cyclically reduced involution. Similarly $b_1 b_2 \cdots_q$ is a reduced word which spells a cyclically reduced involution. By lemma 3.7, each conjugacy class of involution contains a unique cyclically reduced involution. Thus $a_1 a_2 \ldots a_p$ spells the same cyclically reduced involution as $b_1 b_2 \ldots b_q$. So $X = Y$. Therefore i is one-to-one.

Now we must show that i is onto. Let u^W be a conjugacy class of involutions where u is the unique cyclically reduced involution in that conjugacy class. By Lemma 3.5 and Lemma 3.6, u may be may be spelled $a_1 a_2 \ldots a_p$ where the letters are distinct and pairwise commute. Thus there exists a subset of the generators that pairwise commute. Let $X = \{a_1, a_2, \ldots, a_p\}$. Since all the letters in $\{a_1, a_2, \ldots, a_p\}$ pairwise commute, and vertices in Γ are adjacent if and only if they commute, X is a complete subgraph of Γ and $i(X) = u^W$. □

The next two lemmas assert that i induces a graph isomorphism from Ω to Δ.

Lemma 4.2. *Let $X, Y \in V_\Omega$. If $\{X, Y\} \in E_\Omega$ then $\{i(X), i(Y)\} \in E_\Delta$.*

Proof. Assume $\{X, Y\} \in E_\Omega$. Then X and Y are complete subgraphs of Γ and $X, Y \subseteq Z$ for some $Z \in V_\Gamma$. Let $\{a_1, a_2, \ldots, a_p\}$ be the vertex set of X and $\{b_1, b_2, \ldots, b_q\}$ the vertex set of Y. Then $i(X) = (a_1 a_2 \ldots a_p)^W$ and $i(Y) = (b_1 b_2 \ldots b_q)^W$. Since $X \subseteq Z$ and $Y \subseteq Z$, each $a_i \in X$ is in Z and each $b_i \in Y$ is in Z. Since Z is a complete subgraph, the letters $\{a_1, a_2, \ldots, a_p, b_1, b_2, \ldots, b_q\}$ pairwise commute. Thus,

$$a_1 a_2 \ldots a_p b_1 b_1 \ldots b_q = b_1 b_2 \ldots b_q a_1 a_2 \ldots a_p$$

By construction $\{i(X), i(Y)\} \in E_\Delta$. □

Lemma 4.3. *Let $X, Y \in V_\Omega$. If $\{i(X), i(Y)\} \in E_\Delta$ then $\{X, Y\} \in E_\Omega$.*

Proof. Assume $\{i(X), i(Y)\} \in E_\Delta$. By construction $i(X)$ and $i(Y)$ are distinct conjugacy classes of involutions with $u \in i(X)$ and $v \in i(Y)$ such that $uv = vu$. Note that the subgroup generated by u and v is finite (it only contains u, v, uv and 1). By Theorem 3.1, there exists a complete subgraph $Z \subseteq \Gamma$ and a group element $w \in W(\Gamma)$, such that $wGw^{-1} = \{wgw^{-1}; g \in G\} = W(X)$. So wuw^{-1} and wvw^{-1} are elements of $W(Z)$. Since Z is a complete subgraph all the vertices of $W(Z)$ pairwise commute. It follows that all the elements in $W(Z)$, which include wuw^{-1} and wvw^{-1}, are cyclically reduced involutions. By Lemma 3.7, $wuw^{-1} = (\prod_{a_i \in X} a_i)$ and $wvw^{-1} = (\prod_{b_i \in Y} b_i)$. So the letters in $(\prod_{a_i \in X} a_i)$ and $(\prod_{b_i \in Y} b_i)$ are contained in $(\prod_{c_i \in Z} c_i)$. Therefore $X, Y \subseteq Z$. By definition $\{X, Y\} \in E_\Omega$. □

By Lemma 4.2 if there is an edge in Ω there is an edge between the corresponding vertices in Δ. And by lemma 4.3 if there is an edge in Δ then there is an edge between the corresponding vertices in Ω. Thus we have proved the following.

Proposition 1. *The map i induces an isomorphism from Ω to Δ.*

5. Ω Determines Γ

In this section we reconstruct Γ from Ω, thereby inverting the construction of Ω. This establishes that the construction of the graph Ω from Γ is one-to-one. Concatenating the construction of Δ from W, with the isomorphism from Δ to Ω and the reconstruction of Γ, shows that the construction of W is invertible and our theorem is proved..

$$\begin{array}{ccc} \Gamma & \rightarrow & W \\ \updownarrow & & \downarrow \\ \Omega & \cong & \Delta \end{array}$$

5.1. **More Properties of Graphs.** In this section we state and prove a lemma about properties of graphs which are of special consideration to the reconstruction of Γ.

Let $\Gamma = (V_\Gamma, E_\Gamma)$ be a graph.

Definition 5.1. *For each vertex $v_i \in V$. The* **star** *of v_i is*

$$star(v_i) = \{v_j \in V : \{v_i, v_j\} \in E\} \cup \{v_i\};$$

that is, the star of a vertex is the set of all adjacent vertices and the vertex itself.

We define an *equivalence relation* on V_Γ as follows: $v_i \sim v_j$ if and only if $star(v_i) = star(v_j)$. An *equivalence relation* is a commutative, transitive and reflexive relation that partitions a set. We write $[v_i]$ for the set of vertices equivalent to v_i.

Definition 5.2. *Given a graph Γ define a quotient graph $\tilde{\Gamma}$ as follows:*

- *$\tilde{V_\Gamma}$ is the set of equivalence classes of V_Γ.*
- *$\{[v_i], [v_j]\} \in \tilde{E_\Gamma}$ if and only if $\{v_i, v_j\} \in E_\Gamma$.*

Note that the the construction of edges is well-defined; if $u_i \in [v_i]$ and $u_j \in [v_j]$ then $\{u_i, u_v\} \in E_\Gamma \Leftrightarrow \{v_i, v_j\} \in E_\Gamma$.

For any graph Γ the quotient graph $\tilde{\Gamma}$ is a graph invariant property. This means that isomorphic graphs will have isomorphic quotient graphs. In fact, given another grpha Δ, there is an isomorphism $g : V_\Gamma \to V_\Delta$ if and only if $\tilde{\Gamma}$ and $\tilde{\Delta}$ are isomorphic and there is a function $f : V_{\tilde{\Delta}} \to \mathbb{N}$ such that for each v_i, the order of $[v_i]$ is the same as $f([v_i])$.

Lemma 5.3. *A Graph Γ can be determined from $\tilde{\Gamma}$ and a function $f : V_{\tilde{\Gamma}} \to \mathbb{N}$ which sends a vertex $[U] \in V_{\tilde{\Gamma}}$ to the number of elements in $[U]$.*

Proof. The construction proceeds as follows:

- Replace each vertex $[U] \in V_{\tilde{\Gamma}}$ by a complete subgraph containing $f([U])$ vertices, which we label $u_1, u_2, \ldots, u_{f([U])}$.
- If two vertices $[U]$ and $[V]$ are adjacent in $\tilde{\Gamma}$, then every vertex u_i is adjacent to every vertex v_j.

This procedure replaced each vertex in $\tilde{\Gamma}$ with the number of vertices in the corresponding equivalence class. Since the procedure will determine a graph with the same quotient graph as Γ, it determines Γ. □

5.2. **$\tilde{\Gamma}$ and $\tilde{\Omega}$.** Recall that V_Ω is the set of complete subgraphs of Γ. The fact that each vertex in V_Γ is itself a complete subgraph of Γ and is thus also a vertex in Ω, gives a natural injective map $j : V_\Gamma \to V_\Omega$, defined by $j(v_i) = \{v_i\}$.

Lemma 5.4. *j is an injective map that preserves adjacency.*

Proof. Let v_i, v_j be distinct elements in V_Γ.

Case 1: Suppose $\{v_i, v_j\} \in E_\Gamma$. Then $X = \{v_i, v_j\}$ is a complete subgraph of Γ. Since $j(v_i) = \{v_i\}$ and $j(v_j) = \{v_j\}$ are both contained in X, $\{j(v_i), j(v_j)\} \in E_\Omega$.

Case 2. Suppose $\{v_i, v_j\} \notin E_\Gamma$. Then any subgraph of Γ that contains both v_i and v_j will not be complete. Thus $\{j(v_i), j(v_j)\} \notin E_\Omega$.

□

Lemma 5.5. *j induces an injective map $\tilde{j} : \tilde{\Gamma} \to \tilde{\Omega}$ which preserves adjacency.*

Proof. Given $[v_i] \in V_{\tilde{\Gamma}}$ we define $\tilde{j}([v_i]) = [j(v_i)]$.

First we check that $\tilde{j}$ is well-defined. Suppose $[v_i] = [v_j]$; that is v_i and v_j are equivalent vertices in Γ. By definition, $star(v_i) = star(v_j)$. So if v_i is contained in a complete subgraph of Z of Γ then $Z \cup \{v_j\}$ is a complete subgraph. It follows that if $j(v_i)$ is adjacent to a vertex in Ω, $j(v_j)$ is also adjacent to that vertex. By symmetry any vertex adjacent to $j(v_i)$ is also adjacent to $j(v_j)$. Therefore $star(j(v_i)) = star(j(v_j))$ and $\tilde{j}([v_i]) = \tilde{j}([v_j])$.

Now we show that $\tilde{j}$ is injective. Let $[v_i]$ and $[v_j]$ be equivalence classes in Γ with $\tilde{j}([v_i]) = \tilde{j}([v_j])$. Suppose that v_i is adjacent to a vertex v_k in Γ. By Lemma 5.4, $j(v_i)$ is adjacent to $j(v_k)$ in Ω. By hypothesis, $[j(v_i)] = [j(v_j)]$. So $j(v_j)$ is also adjacent to $j(v_k)$. By definition of Ω, $X = \{v_j, v_k\}$ is a complete subgraph in Γ, so v_j is adjacent to v_k. By symmetry if v_j is a adjacent to a vertex v_k in Γ then v_i is adjacent to v_k as well. Therefore the $star(v_i) = star(v_j)$ and $[v_i] = [v_j]$, so $\tilde{j}$ is injective.

Now we show that $\tilde{j}$ preserves adjacency. Let $[v_i], [v_j]$ be distinct equivalence classes in Γ with $\{[v_i], [v_j]\} \in E_{\tilde{\Gamma}}$. By definition, v_i and v_j are adjacent in Γ. By Lemma 5.4, $j(v_i)$ and $j(v_j)$ are adjacent in Ω. Since $\tilde{j}$ is injective, $[j(v_i)]$ and $[j(v_j)]$ are distinct and by definition they are adjacent. Thus $\{\tilde{j}([v_i]), \tilde{j}([v_j])\} \in E_{\tilde{\Omega}}$.

Let $[v_i], [v_j]$ be distinct equivalence classes in Γ with $\{[v_i], [v_j]\} \notin E_{\tilde{\Omega}}$. By definition, v_i and v_j are not adjacent in Γ. By Lemma 5.4, $j(v_i)$ and $j(v_j)$ are not adjacent in Ω. Since $\tilde{j}$ is injective, $[j(v_i)]$ and $[j(v_j)]$ are distinct and by definition they are not adjacent. Thus $\{\tilde{j}([v_i]), \tilde{j}([v_j])\} \notin E_{\tilde{\Omega}}$. $\square$

5.3. **The Procedure for Determining Γ from Ω.** In Lemma 5.5 we exhibited an injective map from $\tilde{\Gamma} \to \tilde{\Omega}$ which preserves adjacency. Since $\tilde{\Gamma}$ is isomorphic to a subgraph of $\tilde{\Omega}$, by Lemma 5.3 we know that we can determine Γ from $\tilde{\Omega}$ with a function $f : V_{\tilde{\Omega}} \to \mathbb{N}$ such that $f([U]) = 0$ if $[U] \cap j(V_\Gamma) = \emptyset$ and $f([U]) = n$ if $|[U] \cap j(V_\Gamma)| = n$. We now define such a function f.

In any graph Γ, for vertices v_i and v_j we write $[v_i] \preceq [v_j]$ if the $star(v_i) \subseteq star(v_j)$ and $[v_i] \prec [v_j]$ if $star([v_i]) \subset star([v_j])$.

Definition 5.6. *We define the* **height**, *$h(v_i)$, to be the length of the maximal strictly ascending chain of stars that starts with v_i. If $star(v_i)$ isn't strictly contained in the star of any other vertex, then we say that it has height 1.*

Definition 5.7. *Let Γ be a graph. For a vertex $v_i \in V_\Gamma$, we define θ_{v_i} to be the union of all vertices $v_j \in \Gamma$, $star(v_i) \preceq star(v_j)$. We write $|\theta_{v_i}|$ for the number of vertices in θ_{v_i}.*

Recall that each element of $V_{\tilde{\Omega}}$ is an equivalence class of vertices in Ω. We define f as follows:

$$f : V_{\tilde{\Omega}} \to \mathbb{N}$$
$$f([U]) = log_2(|\theta_{[U]}| + 1) - \sum_{[U] \prec [V]} f([V])$$

Using this function we use the construction first described in Lemma 5.3 to determine Γ.

- Replace each vertex $[U]$ in $\tilde{\Omega}$ by a complete subgraph containing $f([U])$ vertices, which we label $v_1, v_2, \ldots, v_{f([U])}$.
- If two $[U]$ and $[V]$ are adjacent in $\tilde{\Omega}$, then every vertex u_i is adjacent to every vertex v_j.

Recall that $\theta_{[U]}$ is the union of all complete subgraphs of V_Ω whose star contains the star of U.

Lemma 5.8. *For each $[U] \in V_{\tilde{\Omega}}$, $log_2(|\theta_{[U]}| + 1) = |\theta_{[U]} \cap j(V_\Gamma)|$.*

Proof. We prove the lemma via 3 claims.

Claim 1: There is a unique complete subgraph maximal to $\theta_{[U]}$. By maximal we mean that there is no larger complete subgraph in $\theta_{[U]}$.

Since Γ is finite, there exists a complete subgraph X in $\theta_{[U]}$ such that $|X| = m$ and $m = max\{|Y| : Y \in \theta_{[U]}, Y \in V_\Omega\}$. Suppose that there exist two distinct complete subgraphs $X, Y \in \theta_{[U]}$ where $|X| = |Y| = m$. There is some vertex $a_i \in X, a_i \notin Y$. So consider $Y \cup \{a_i\}$. Since $a_i \in X$, and $[X] \succeq [U]$, $[a_i] \succeq [U]$. Thus $Y \cup \{a_i\} \in V_\Omega$ and $star(Y \cup \{a_i\}) \supseteq star(U)$. Since $|Y \cup \{a_i\}| = m + 1$, this contradicts the fact that $|Y|$ was maximal.

Lets X be the largest complete subgraph in $\theta_{[U]}$ and let $m = |X|$.

Claim 2: If $Z \subset X$, then $Z \in \theta_{[U]}$. Let $Z \subset X$. Then every complete subgraph containing X contains Z. Thus $[X] \preceq [Z]$. So by definition $Z \in \theta_{[U]}$.

Claim 3: If Z is not contained in X, then $Z \notin \theta_{[U]}$.

Let Z be a complete subgraph that is not a subset of X. So there exists an $a_i \in Z$ with $a_i \in X$. Suppose that $Z \in theta_{[U]}$, then $star(Z) \supseteq star(U)$. Consider $X \cup \{a_i\}$. Since $a_i \in Z$, and $[Z] \succeq [U]$, $[a_i] \succeq [U]$. Thus $X \cup \{a_i\} \in V_\Omega$ and $star(X \cup \{a_i\}) \supseteq star(U)$. Since $|X \cup \{a_i\}| = m + 1$, this contradicts the fact that $|X|$ was maximal. Thus $Z \notin \theta_{[U]}$.

Now consider how many complete subgraphs are in $\theta_{[U]}$. There are m vertices in X. By claim 3, only subsets of X will be in $\theta_{[v_i]}$. By claim 2, every subset of X will be in $\theta_{[U]}$. So $\theta_{[U]}$ contains $2^m - 1$ complete subgraphs, m of which are singleton complete subgraphs. □

Now we use this lemma inductively to show that $f([j(v_i)])$ is the number of elements in $[v_i]$.

Lemma 5.9. *For each* $[U] \in V_{\tilde{\Omega}}$, $f([U]) = |[U] \cap j(V_\Gamma)|$.

Proof. We proceed by induction on the height of equivalence classes.

Let $h([U]) = 1$. Then $\theta_{[U]} = [U]$. By Lemma 5.8, $f([U])$ is the number of singleton complete subgraphs in $[U]$.

Assume that $f([U])$ correctly determines the number of singleton complete subgraphs for each $[U]$ where $1 \leq h([U]) \leq n$. Suppose $[U]$ has height $n + 1$. We defined $f([U])$ as $log_2(|\theta_{[U]}| + 1) - \sum_{[U] \prec [V]} f([V])$. By Lemma 5.8, $log_2(|\theta_{[U]}| + 1)$ is the number of singleton complete subgraphs whose $star$ contains the $star(U)$. By hypothesis, $\sum_{[U] \prec [V]} f([V])$ is the number of singleton complete subgraphs whose star strictly contains the star of U. It follows that $log_2(|\theta_{[U]}|+1) - \sum_{[U]\prec[V]} f([V])$ is the number of singleton complete subgraphs in $[U]$. $\square$

Proposition 2. *Let Γ be a graph and Ω be the graph constructed from Γ as in Section 4. Then Ω determines the isomorphism type of Γ.*

Proof. In Lemma 5.5 we showed that $\tilde{\Gamma} \hookrightarrow \tilde{\Omega}$. In Lemma 5.9 we showed that for $[U] \in V_{\tilde{\Omega}}$, $f([U])$ is the order of $j(\Gamma) \cap [U]$. Thus by Lemma 5.3, Ω determines Γ. $\square$

6. An Example

We examine an example to elucidate the construction of Ω and it's inverse. Let Γ be as in Figure 5.

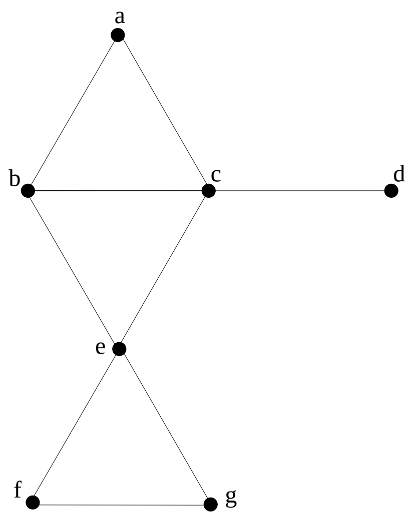

Figure 5. Example Γ.

We construct Ω. To do this we must determine all the complete subgraphs of Γ. Then we must determine which pairs of complete subgraphs are both subgraphs of a complete subgraph. The set of complete subgraphs is:

$$\{a, b, c, d, e, f, g, ab, ac, bc, be, cd, ce, ef, eg, fg, abc, bce, efg\}$$

. Since the complete subgraphs a, b, c, ab, bc, ac, abc are all subgraphs of abc, they are pairwise adjacent in Ω. This is illustrated by the triangle with corners a, b and c in Figure 6. The rest of Ω is constructed similarly.

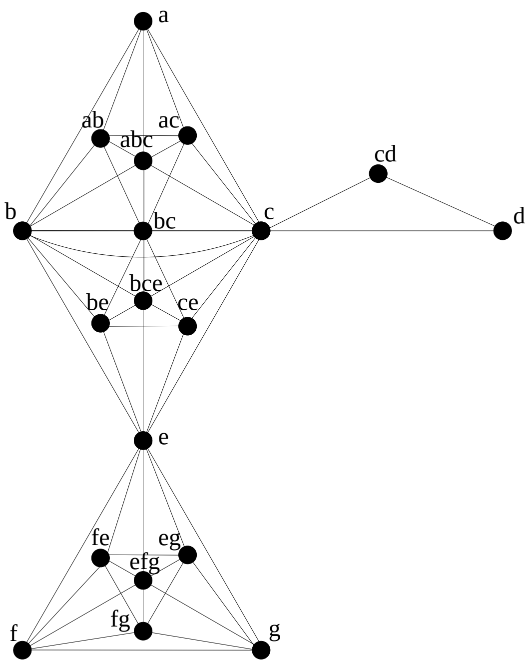

FIGURE 6. Example Ω.

Next we determine which vertices share the same star. In Figure 7 we have circled all the vertices in Ω that share the same star. With this information we can then build $\tilde{\Omega}$. We insert a vertex for each circle, then connect two vertices if elements contained in the corresponding circles are adjacent(see Figure 8).

At this point we our ready to use the procedure to reconstruct Γ. To determine Γ we need to find the height, $\theta_{[v_i]}$, and calculate $f([v_i])$ for each equivalence class $[v_i]$. For example consider $[a]$. Let's determine all the strictly ascending chains starting with $[a]$: we have $[a], [a] \prec [c]$ and $[a] \prec [b] \prec [c]$. Thus $[a]$ has height 3. Now we determine $\theta_{[a]}$. It is the set $\{a, b, c, ab, ac, bc, abc\}$ since those are the vertices in $[a], [b]$ and $[c]$. Note $[c]$ has height one and there is only one vertex in $\theta_{[c]}$ so $f([c]) = 1$. We also note that $[b]$ has height two, $\theta_{[c]} = \{b, c, bc\}$. So $f([b]) = 1$. Finally we compute:

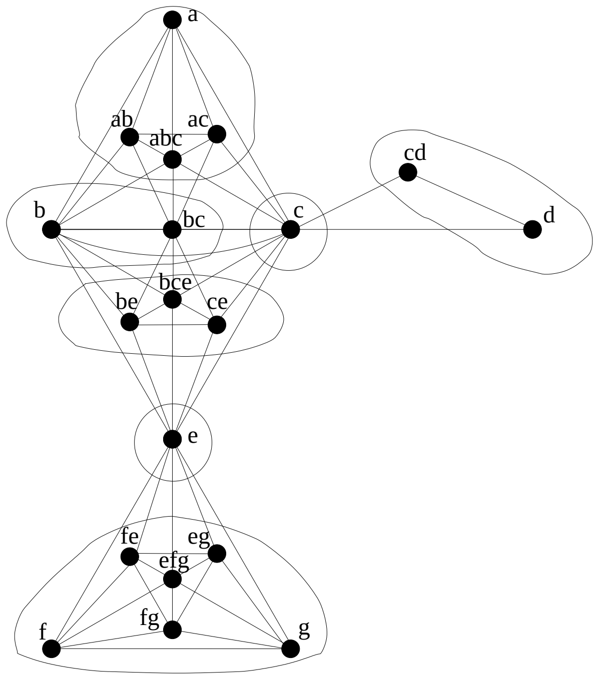

FIGURE 7. Example Ω with Equivalence Classes Circled.

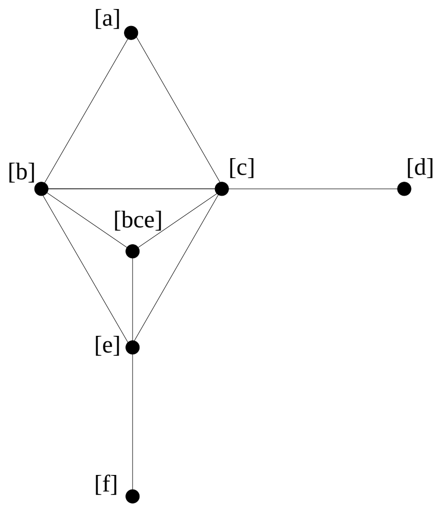

FIGURE 8. Example $\tilde{\Omega}$.

$$\begin{aligned} f([a]) &= log_2(|\theta_{[a]}|+1) - \sum_{[v_i]\prec[a]} f([v_j]) \\ &= log_2(|7|+1) - (f([b]) + f([c])) \\ &= 3 - (1+1) \\ &= 1. \end{aligned}$$

We make similar calculations for all the other $[v_i]$.

$[v_i]$	$h(v_i)$	$\theta_{[v_i]}$	$\lvert\theta_{[v_i]}\rvert$	$log_2(\lvert\theta_{[v_i]}\rvert + 1)$	$\sum_{[v_i]\prec[v_j]} f([v_j])$	$f([v_i])$
$[a]$	3	$\{a, b, c, ab, ac, bc, abc\}$	7	3	2	1
$[b]$	2	$\{b, bc, c\}$	3	2	1	1
$[c]$	1	$\{c\}$	1	1	0	1
$[d]$	2	$\{d, cd, c\}$	3	2	1	1
$[bce]$	3	$\{b, c, e, be, ce, bc, bce\}$	7	3	3	0
$[e]$	1	$\{e\}$	1	0	1	
$[f]$	2	$\{e, f, g, ef, eg, fg, efg\}$	7	3	1	2

Now we replace each vertex in $\tilde{\Omega}$ with a complete subgraph with $f([v_i])$ vertices. A set of vertices corresponding to $f([v_i])$ is adjacent to vertices corresponding to $f([v_j])$ if and only if $[v_i]$ is adjacent to $[v_j]$. The resultant graph is presented in figure 9. From Figures 5 and 9 it is clear that the reconstructed graph is the original graph Γ

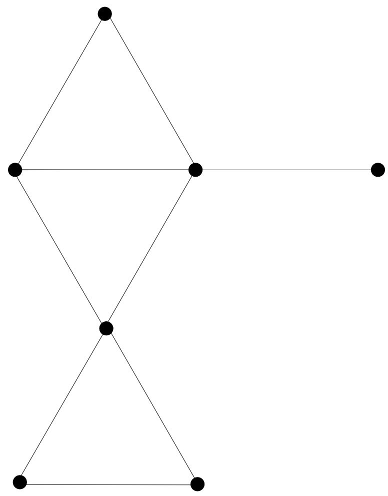

FIGURE 9. Reconstruction of Γ.

7. Future Work

The methods of this proof might be used to prove the rigidity of even Coxeter groups. There are several ideas and methods in this proof that are specific to right-angled Coxeter groups. For example, the deletion conditions is different for different classes of Coxeter groups. However, the basic structure of this proof might extend to even Coxeter groups. In this case even Coxeter groups have a unique associated even labeled graph.

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