# Statistics of the Gauss-Kuzmin Distribution

Steven Duff Advisor: Nathan Ryan
Bucknell University

## Simple Continued Fractions Expansions

Any real number $x$ can be has a unique simple cf expansion:

$$x = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cfrac{1}{a_4 + \ddots}}}} = [a_0, a_1, \dots]$$

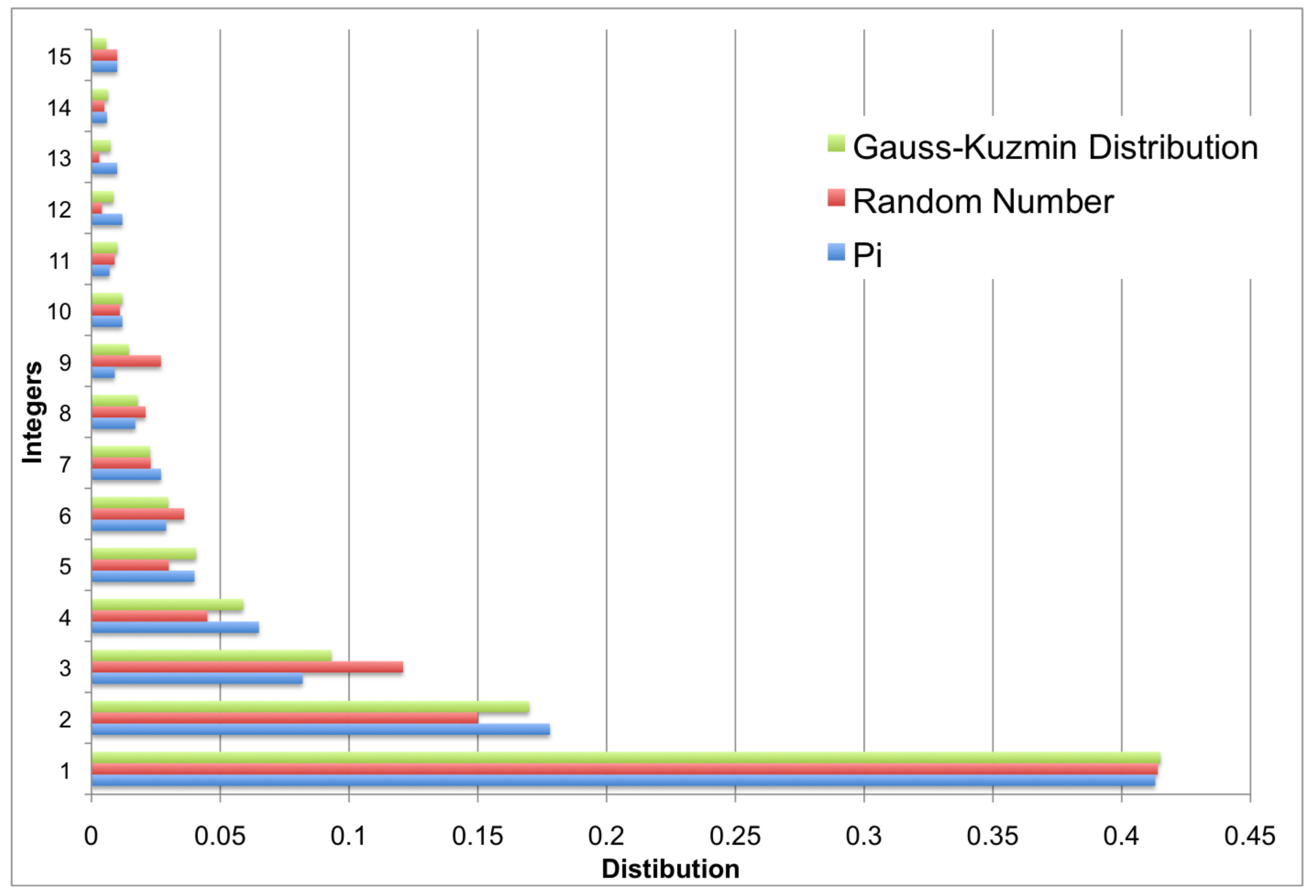
**Terms** $\{a_n\}$**:** Sequence of integers where $a_k > 0$ for $k > 0$.
**Convergents** $\frac{p_n}{q_n}$**:** A quotient equal to a truncated cf expansion (i.e. $\frac{p_n}{q_n} = [a_0, a_1, \dots, a_n]$).
**Infinite cf's:** A cf expansion with an infinite number of terms. $x$ has an infinite cf expansion if and only if it is irrational.

## Gauss-Kuzmin Distribution

For almost all $x \in \mathbb{R}$, $\text{Prob}(a_n = k) = \log_2\left(1 + \frac{1}{k(k+2)}\right)$, with a maximum error term $\frac{A}{k(k+1)} e^{-B\sqrt{n-1}}$, where $A, B$ are positive constants.

This error term approaches 0 as $n$ grows, so the distribution becomes exact in the limit of the terms. This holds for all $x \in K$, where $K \subset \mathbb{R}$ is a set of full measure. There exists an exceptional set $Z$ of measure zero for which this distribution does not hold.

## Distribution of 1-15 in First 1000 Terms of CF Expansions




## Motivation

Not much is known about the exceptional set $Z$, which the Gauss-Kuzmin distribution does not hold for. Finding elements contained in $Z$ will help us characterize $Z$. We developed a framework for experiments that will provide statistical evidence to which set any real $x$ is contained in. To develop this test, we first developed a method to randomly generating numbers that follow Gauss-Kuzmin.

## Some Known Elements of the Exceptional Set $Z$

- **Rational numbers:** Rational numbers have finite cf expansions, and thus can not follow Gauss-Kuzmin for $a_n$ if $n$ is too large.
- **Bounded cf's:** A bounded cf has a maximum term. This means that any integer greater than this maximum term has probability zero of occurring.
- **Quadratic irrationals (i.e.** $\{\frac{a+b\sqrt{c}}{d} | a, b, c, d \in \mathbb{Z}\}$**):** These have periodic cf expansions. This means that their expansions will be bounded and thus they are contained in $Z$.

## Generating Random Continued Fractions

**Original Method:**
Put $r_n := a_n + \frac{1}{a_{n+1} + \frac{1}{\ddots}} = a_n + \frac{1}{r_{n+1}}$
- Randomly choose $x \in (0, 1)$
- $r_0 = x$
- $a_n = \lfloor r_n \rfloor$
- $r_{n+1} = \frac{1}{r_n - a_n}$

**Our Method:**
Put $\frac{p_n'}{q_n'} := [a_0, a_1, \dots, a_n + 1]$
- Randomly choose $x \in (0, 1)$
- $a_0 = \lfloor x \rfloor$
- Randomly choose $x_n \in \left[\frac{p_n}{q_n}, \frac{p_n'}{q_n'}\right]$
- $r_{n+1} = \frac{p_{n-1} - x_n \cdot q_{n-1}}{x_n \cdot q_n - p_n}$
- $a_{n+1} = \lfloor r_{n+1} \rfloor$

The benefit of the new method is that it calls on the random real generator every term, which will increase the randomness of these cf's.

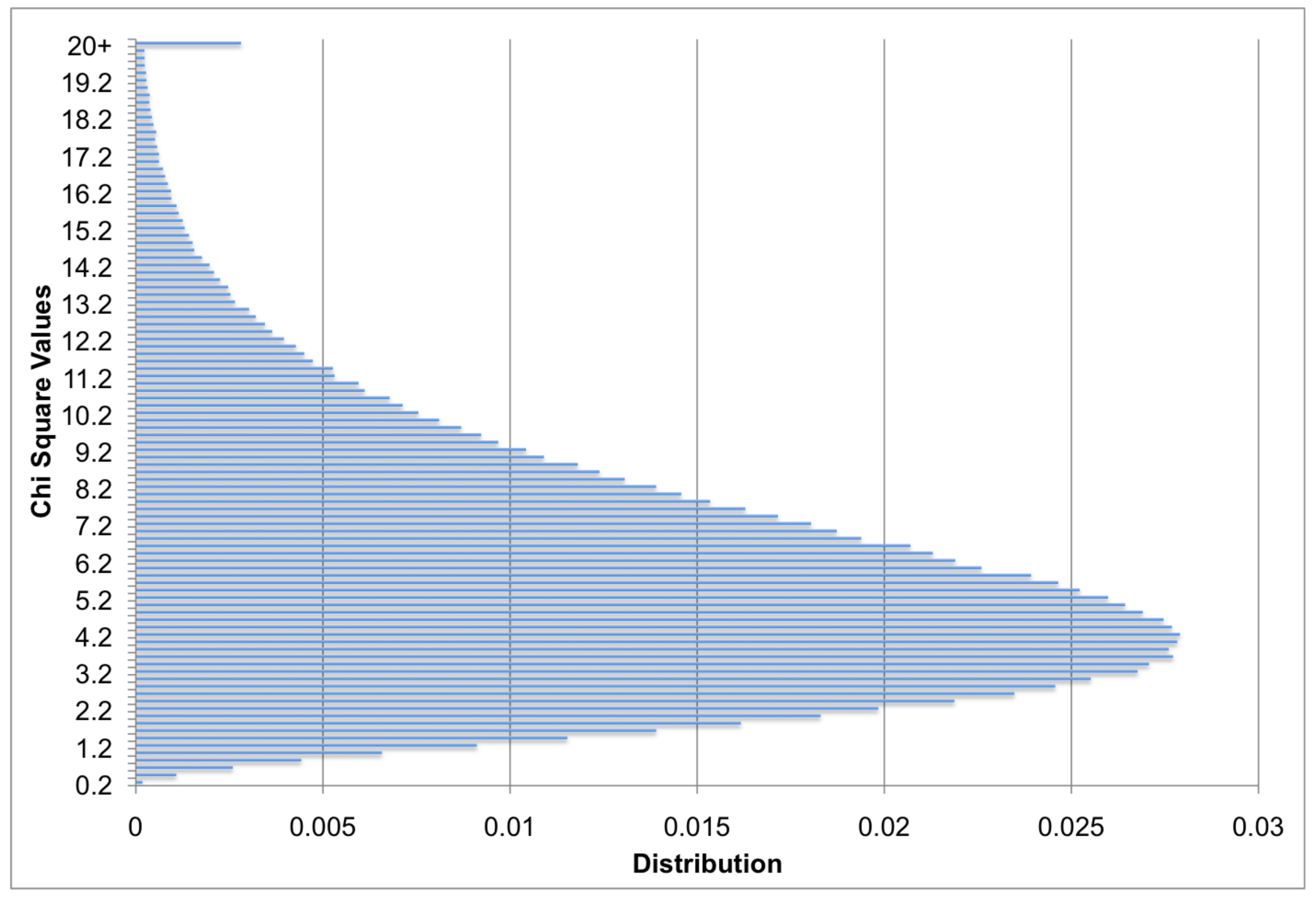
## Experiment

For any real $x$, we want to know whether or not $x$ follows Gauss-Kuzmin.
$H_0 : x \in K$, $H_a : x \in Z$
We will use a $\chi^2$ test to determine this. This test has the form:

$$\chi^2 = \sum_{i=1}^{n} \frac{(O_i - E_i)^2}{E_i}$$

$O_i$ = observed number in $i^{th}$ bin, $E_i$ = expected number in $i^{th}$ bin
**Bins:** $(1), (2), (3), (4), (5), (6-10), (11+)$ **Terms:** $1 - 150$

## $\chi^2$ Values from 1,000,000 Randomly Generated CF's




## Applications

**Question:** Does $e^n$ for positive integers $n$ follow the Gauss Kuzmin Distribution? We will run $\chi^2$ tests and use p-values from the million random cf's we generated.

| | $\chi^2$ | p-value |
|---|---|---|
| $e$ | 118.94 | $< .01$ |
| $e^2$ | 292.385 | $< .01$ |
| $e^3$ | 1.200 | .98 |
| $e^4$ | 8.801 | .17 |
| $e^5$ | 3.541 | .73 |
| $e^6$ | 4.424 | .60 |
| $e^7$ | 2.716 | .84 |
| $e^8$ | 9.579 | .13 |

**Conclusion:** We can say with strong confidence that $e, e^2 \in Z$, but it appears that $e^n$ for $n > 2$ is contained in $K$. The reasons for this could be investigated further.
**Further Questions:** We know that the square roots of integers (quadratic irrationals) do not follow Gauss-Kuzmin. What about other roots of integers? We could investigate if the families of $n^{th}$ roots of integers for $n > 2$ are in $Z$.

## References

- Gauss, Karl Friedrich (1777-1855): Struik, D. J. *A Concise History of Mathematics*. p. 142f.
- Miller, Steven J., and Ramin Takloo-Bighash. *An Invitation to Modern Number Theory*. Princeton: Princeton UP, 2006. Print.
- William Stein. *Sage: Open Source Mathematical Software (Version 4.3.1)*. The Sage Group, 2010