Infinitely many new partition statistics

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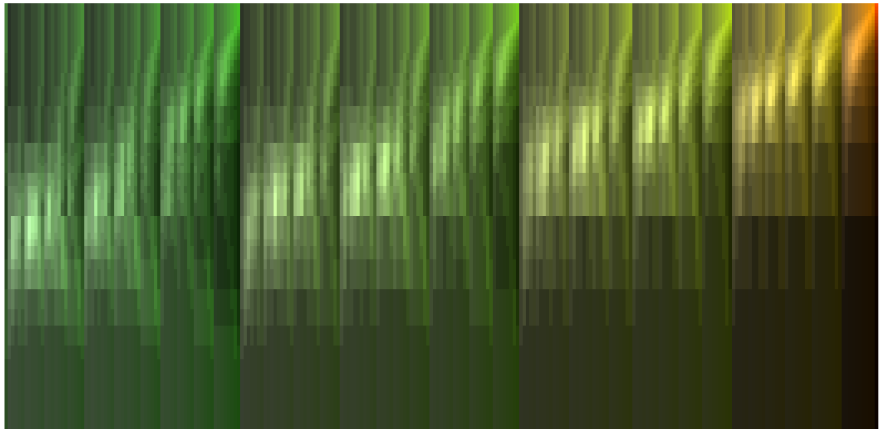


AMS Sectional Meeting, Penn State

October 25, 2009

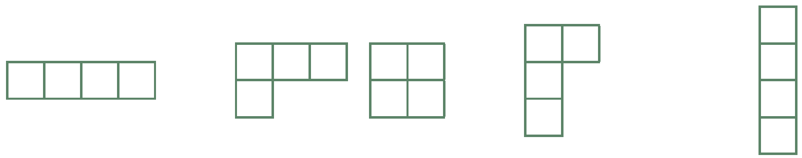
Joint with **Nick Loehr** (Virginia Tech)

$$P(t, q) = \prod_{i \geq 1} \frac{1}{1 - tq^i}$$

$$P(t,q) = \prod_{i\geq 1} \frac{1}{1-tq^i}$$

One interpretation

$[t^k q^n]P(t,q)$ is the number of partitions of n into exactly k parts.

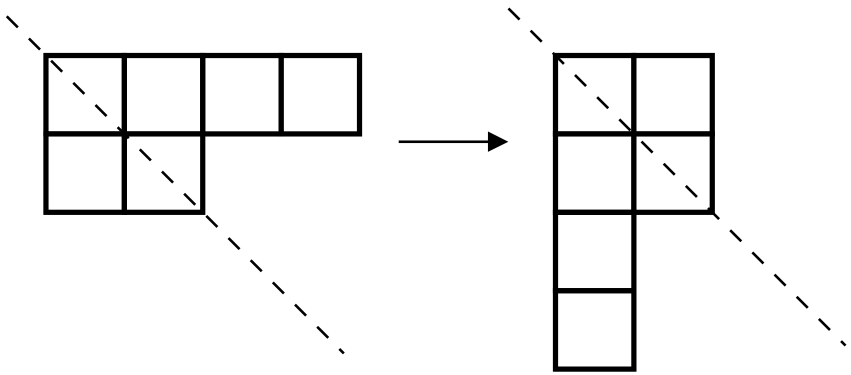


$$P(t,q) = \cdots + \left(t + 2t^2 + t^3 + t^4\right) q^4 + \cdots .$$

$$P(t,q) = \prod_{i\geq 1} \frac{1}{1-tq^i}$$

Another interpretation

$[t^k q^n] P(t,q)$ is the number of partitions of n with largest part of size k.



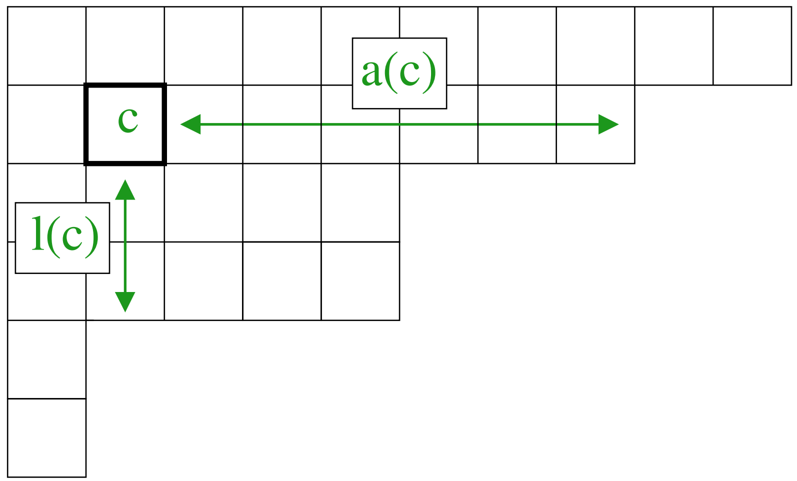
Goal

Complete

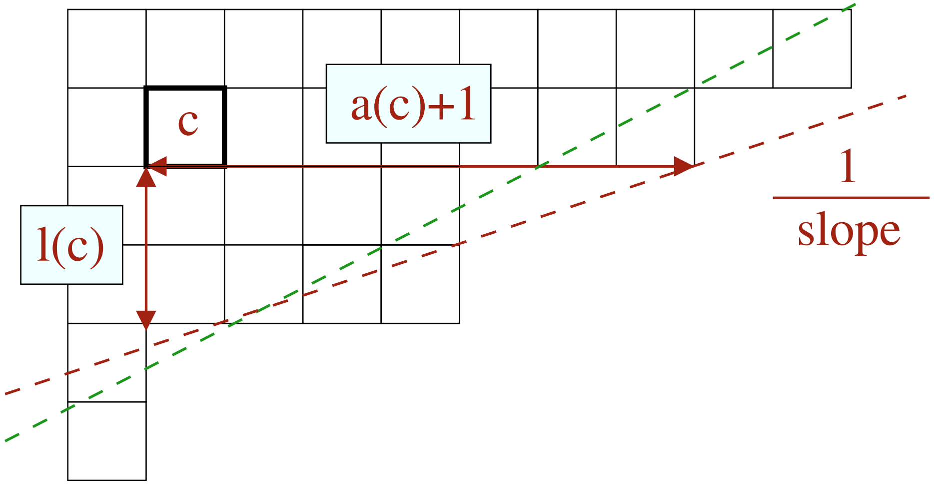
“$[t^k q^n] P(t, q)$ is the number of partitions of n. . . ”

in infinitely many ways.

Head, shoulders, knees & toes



Head, shoulders, knees & toes



$$\frac{1}{\text{slope}} = \frac{a(c)}{l(c)+1}$$

$$\frac{1}{\text{slope}} = \frac{a(c)+1}{l(c)}$$

$h_x^{\pm}(\lambda)$

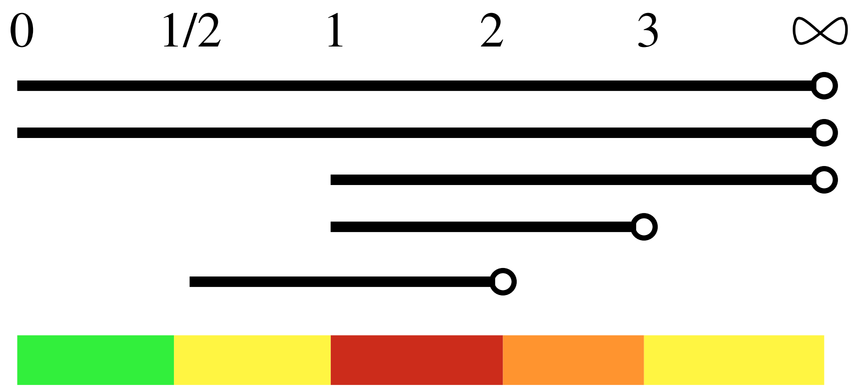
For $x \in [0, \infty)$, $y \in (0, \infty]$,

$$h_x^+(\lambda) = \sum_{c \in \lambda} \chi\left(\frac{a(c)}{l(c)+1} \leq x < \frac{a(c)+1}{l(c)}\right),$$

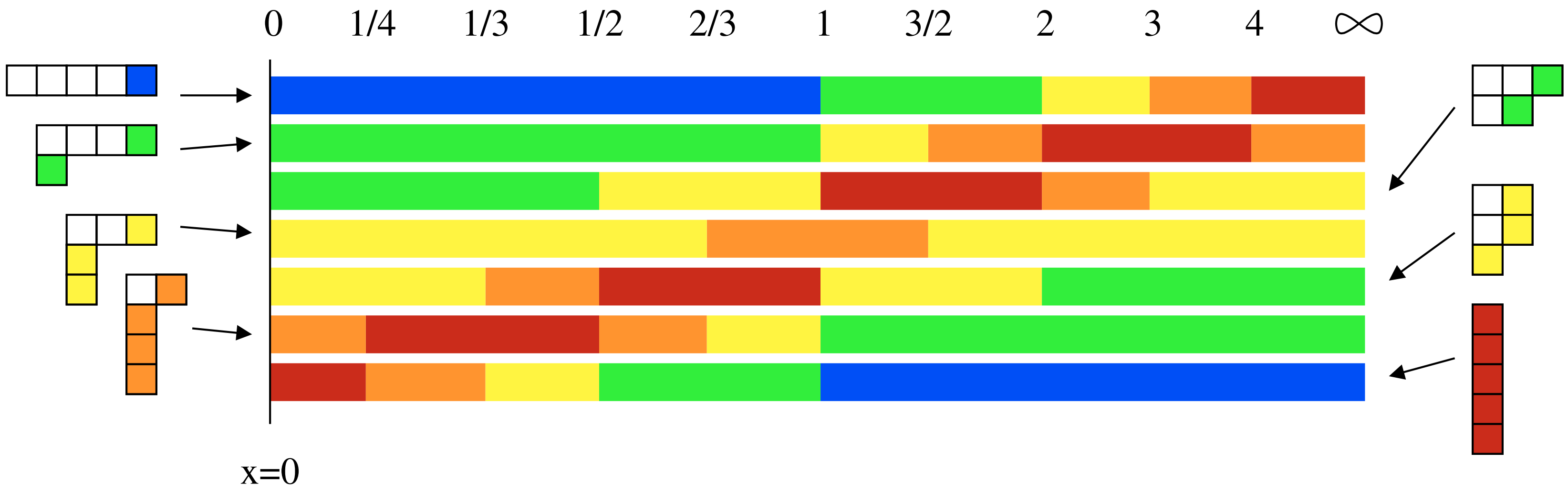
$$h_y^-(\lambda) = \sum_{c \in \lambda} \chi\left(\frac{a(c)}{l(c)+1} < y \leq \frac{a(c)+1}{l(c)}\right).$$

$h_x^+(3, 2)$ example

$[1,3)$	$[1/2,2)$	$[0,\infty)$
$[1,\infty)$	$[0,\infty)$	

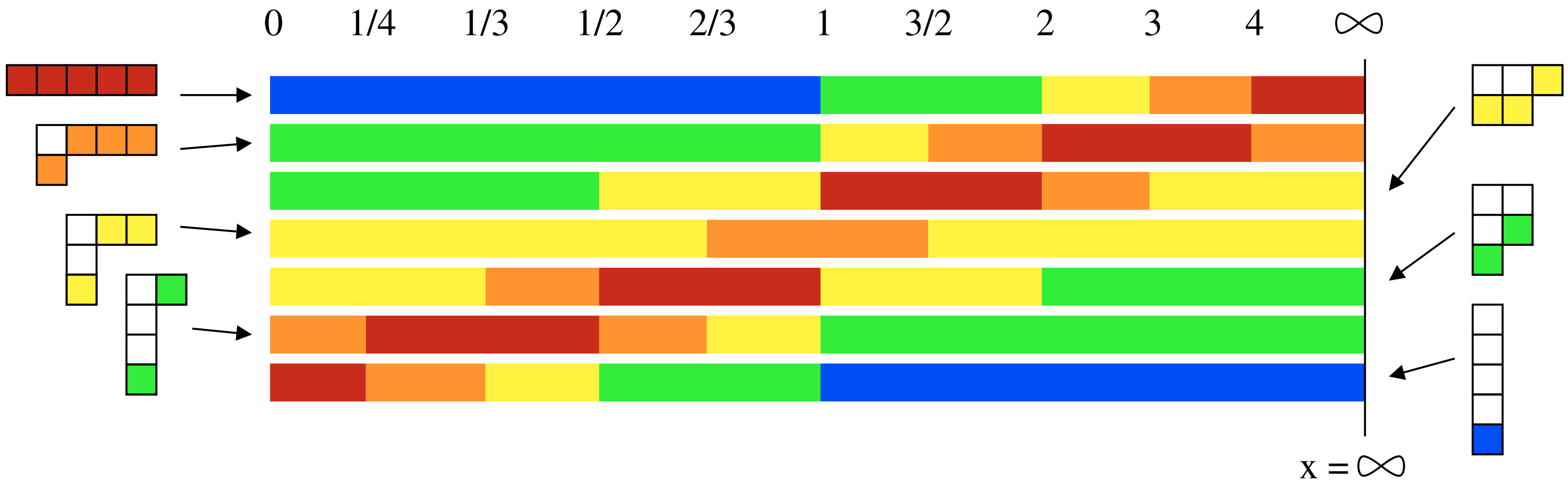


h_0^+ example



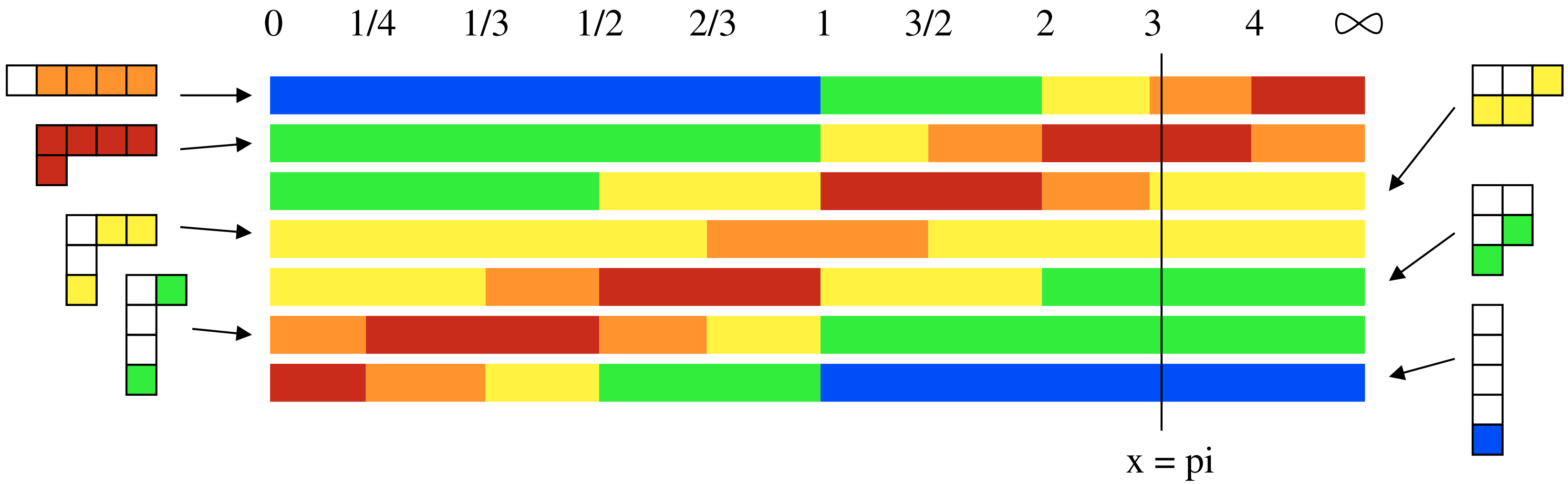
$$\sum_{\lambda \vdash n \geq 0} t^{\ell(\lambda)} q^n = \cdots + (t + 2t^2 + 2t^3 + t^4 + t^5) q^5 + \cdots$$

h_∞^- example



$$\sum_{\lambda \vdash n \geq 0} t^{\lambda_1} q^n = \cdots + (t + 2t^2 + 2t^3 + t^4 + t^5) q^5 + \cdots$$

h_π^+ example



$$\sum_{\lambda \vdash n \geq 0} t^{h_\pi^+(\lambda)} q^n = \cdots + (t + 2t^2 + 2t^3 + t^4 + t^5) q^5 + \cdots$$

Theorem[Haiman (ca. 2000); Loehr-W]

For $x \in [0, \infty)$,

"$[\boldsymbol{t}^k \boldsymbol{q}^n]\boldsymbol{P}(\boldsymbol{t}, \boldsymbol{q})$ is the number

of partitions $\boldsymbol{\lambda} \vdash \boldsymbol{n}$

for which $\boldsymbol{h}_x^+(\boldsymbol{\lambda}) = \boldsymbol{k}$."

Theorem[Haiman (ca. 2000); Loehr-W]

For $x \in [0, \infty)$, $y \in (0, \infty]$,

$$\prod_{i \geq 1} \frac{1}{1 - tq^i} = \sum_{\lambda \vdash n \geq 0} t^{\ell(\lambda)} q^n$$

$$= \sum_{\lambda \vdash n \geq 0} t^{h_x^+(\lambda)} q^n$$

$$= \sum_{\lambda \vdash n \geq 0} t^{h_y^-(\lambda)} q^n.$$

A rephrasing

For $x \in [0, \infty]$, $\delta \in \{+, -\}$,

Define $\boldsymbol{H_x^\delta(n;t) = \sum_{\lambda \vdash n} t^{h_x^\delta(\lambda)}}$.

A rephrasing

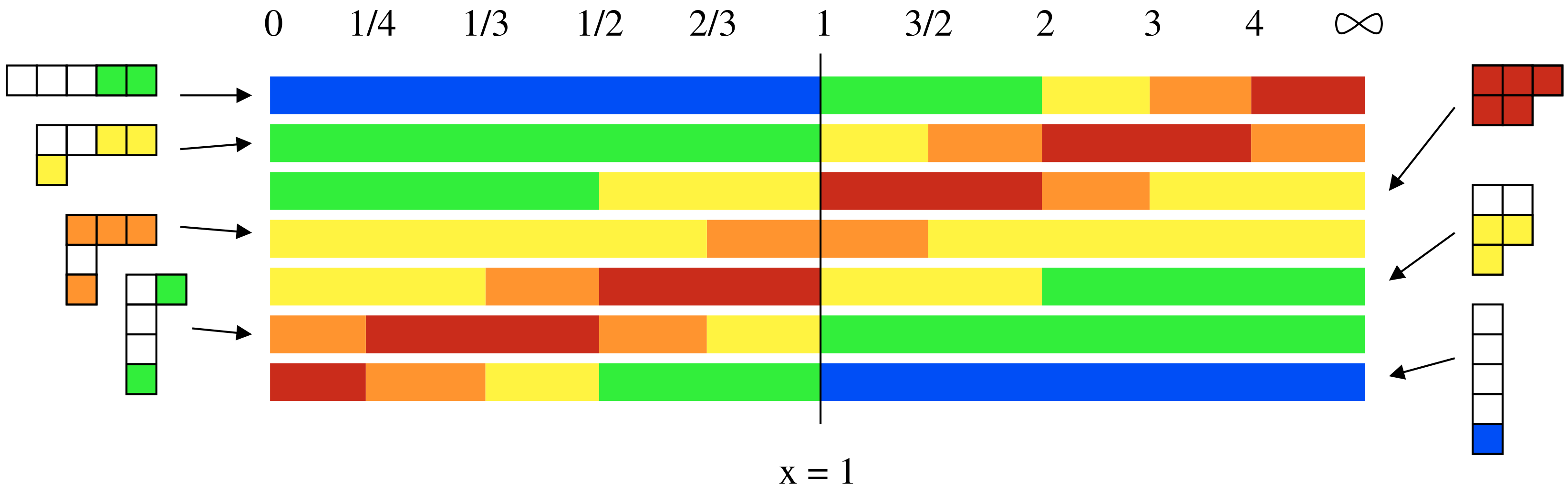
For $x \in [0, \infty]$, $\delta \in \{+, -\}$,

Define $\boldsymbol{H_x^\delta(n;t)} = \sum_{\boldsymbol{\lambda \vdash n}} \boldsymbol{t^{h_x^\delta(\lambda)}}$.

Show For fixed $\boldsymbol{n}$,

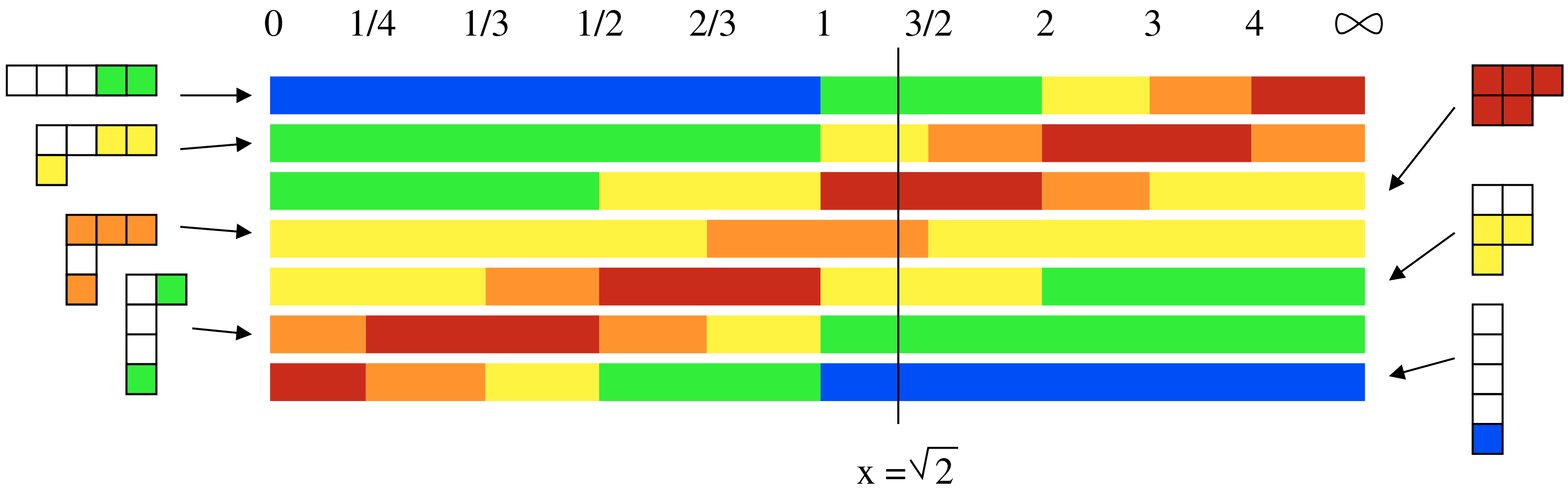
$\boldsymbol{H_x^\delta(n;t)}$ is independent of $\boldsymbol{x}$, $\boldsymbol{\delta}$.

h_1^+ example



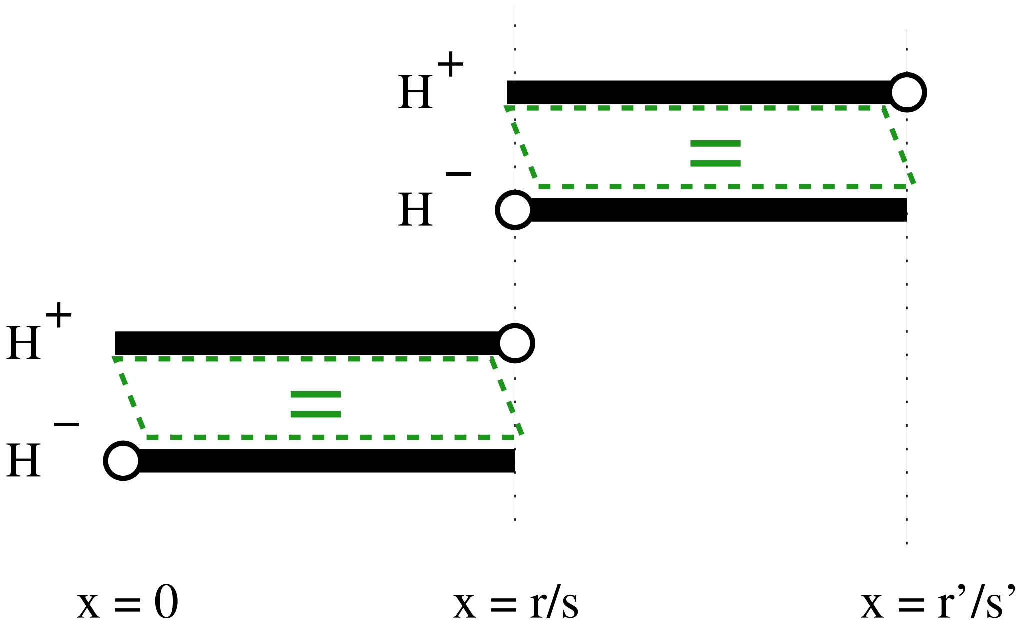
$$\sum_{\lambda \vdash n \geq 0} t^{h_1^+(\lambda)} q^n = \cdots + (t + 2t^2 + 2t^3 + t^4 + t^5) q^5 + \cdots$$

$h^+_{\sqrt{2}}$ example



$$\sum_{\lambda \vdash n \geq 0} t^{h^+_{\sqrt{2}}(\lambda)} q^n = \cdots + (t + 2t^2 + 2t^3 + t^4 + t^5) q^5 + \cdots$$

Proof: A combinatorial homotopy



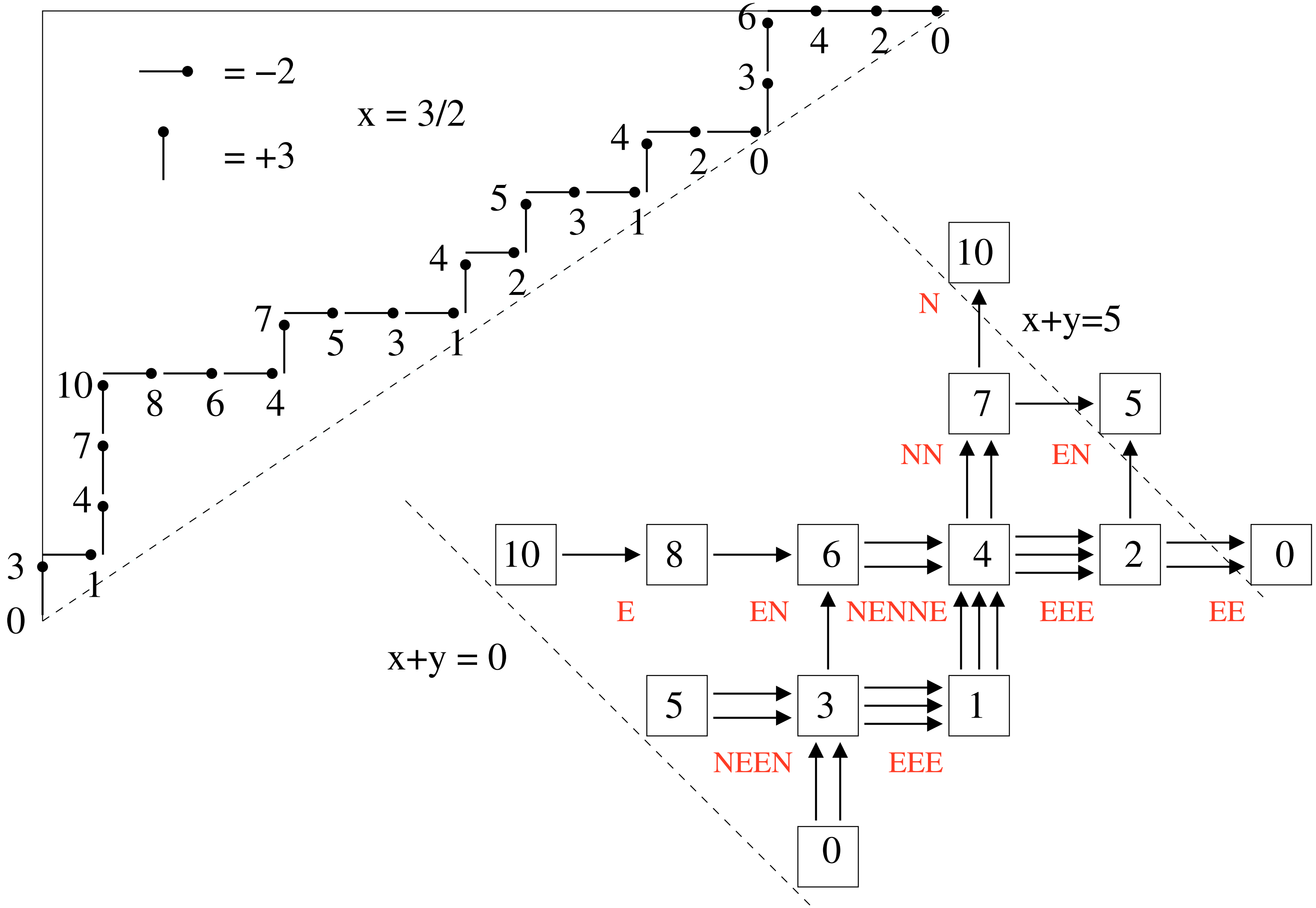
Main Lemma [Loehr-W]

For all positive rational r/s and all integers $n \geq 0$,

$$H^{+}_{r/s}(n;t) = \sum_{\lambda \vdash n} t^{h^{+}_{r/s}(\lambda)}$$

$$= \sum_{\lambda \vdash n} t^{h^{-}_{r/s}(\lambda)} = H^{-}_{r/s}(n;t).$$

à la J. Sjöstrand

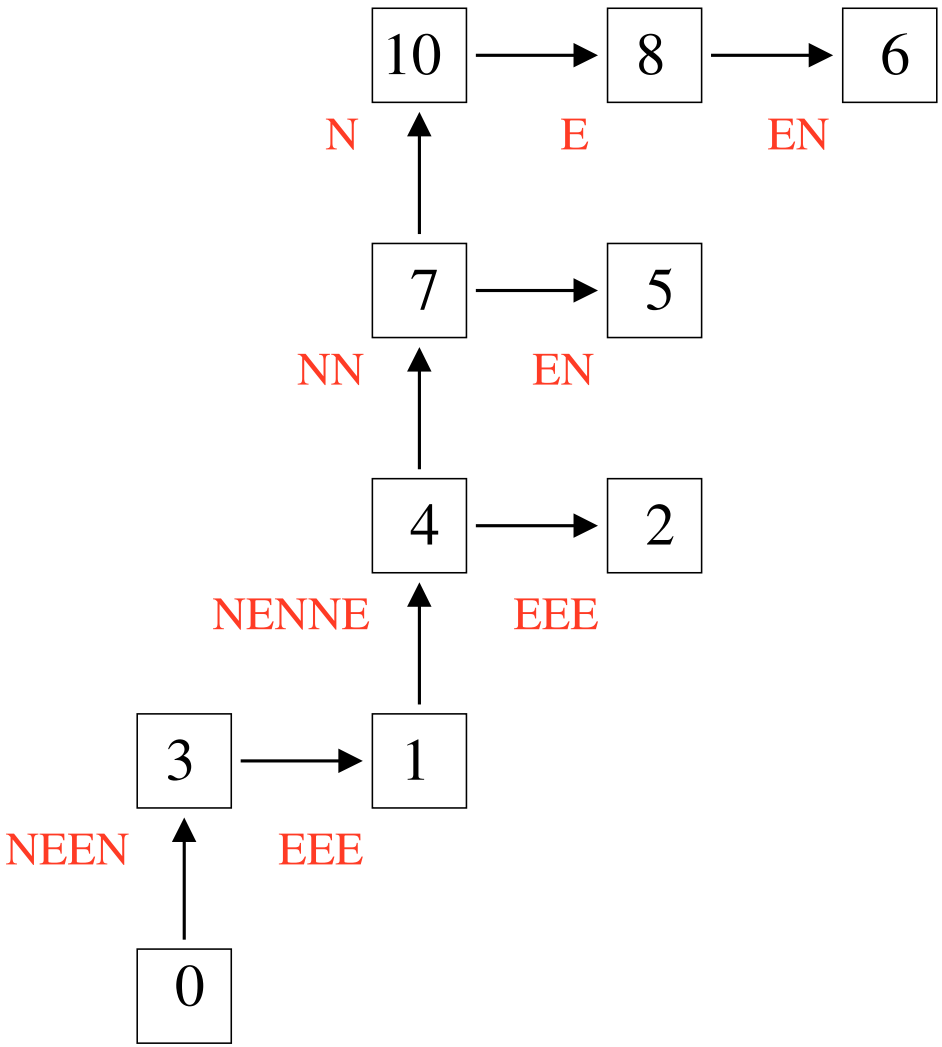


Dependencies

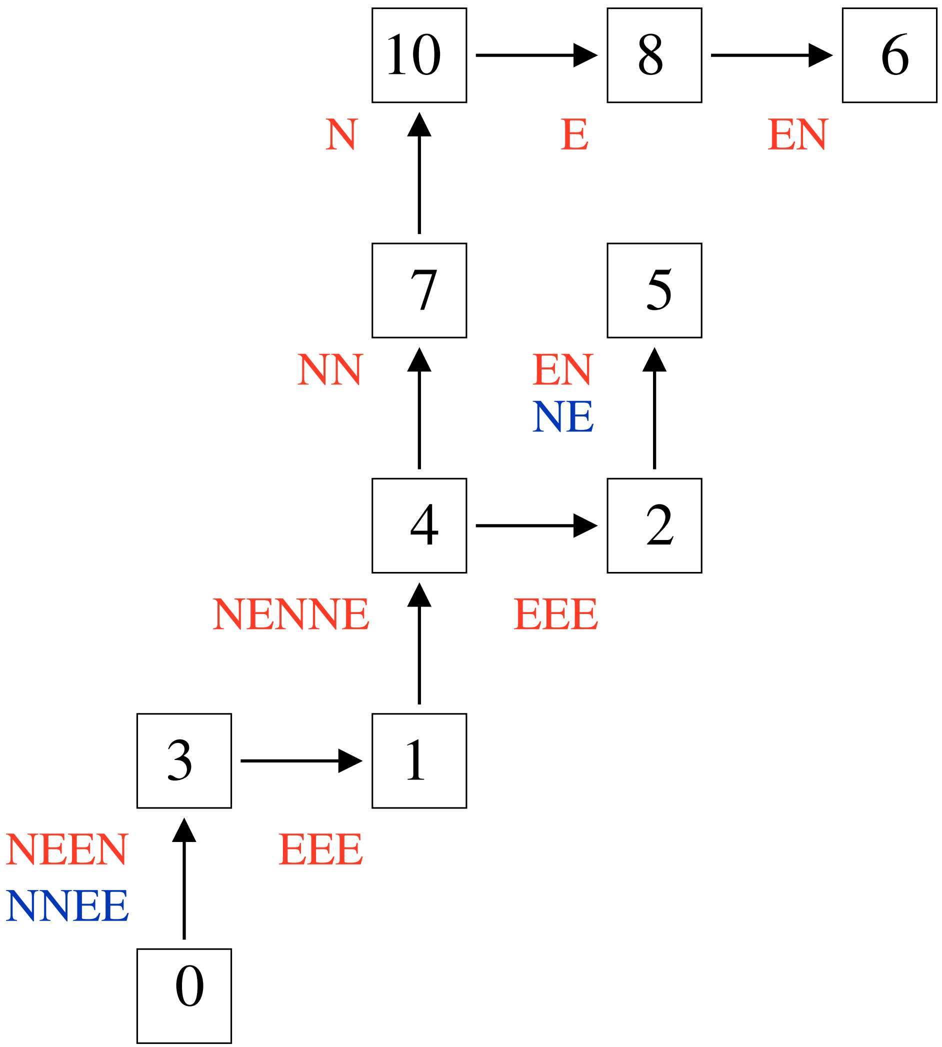
Write $$h^{+}_{r/s} = m_{r/s} + c^{+}_{r/s},$$
$$h^{-}_{r/s} = m_{r/s} + c^{-}_{r/s}.$$

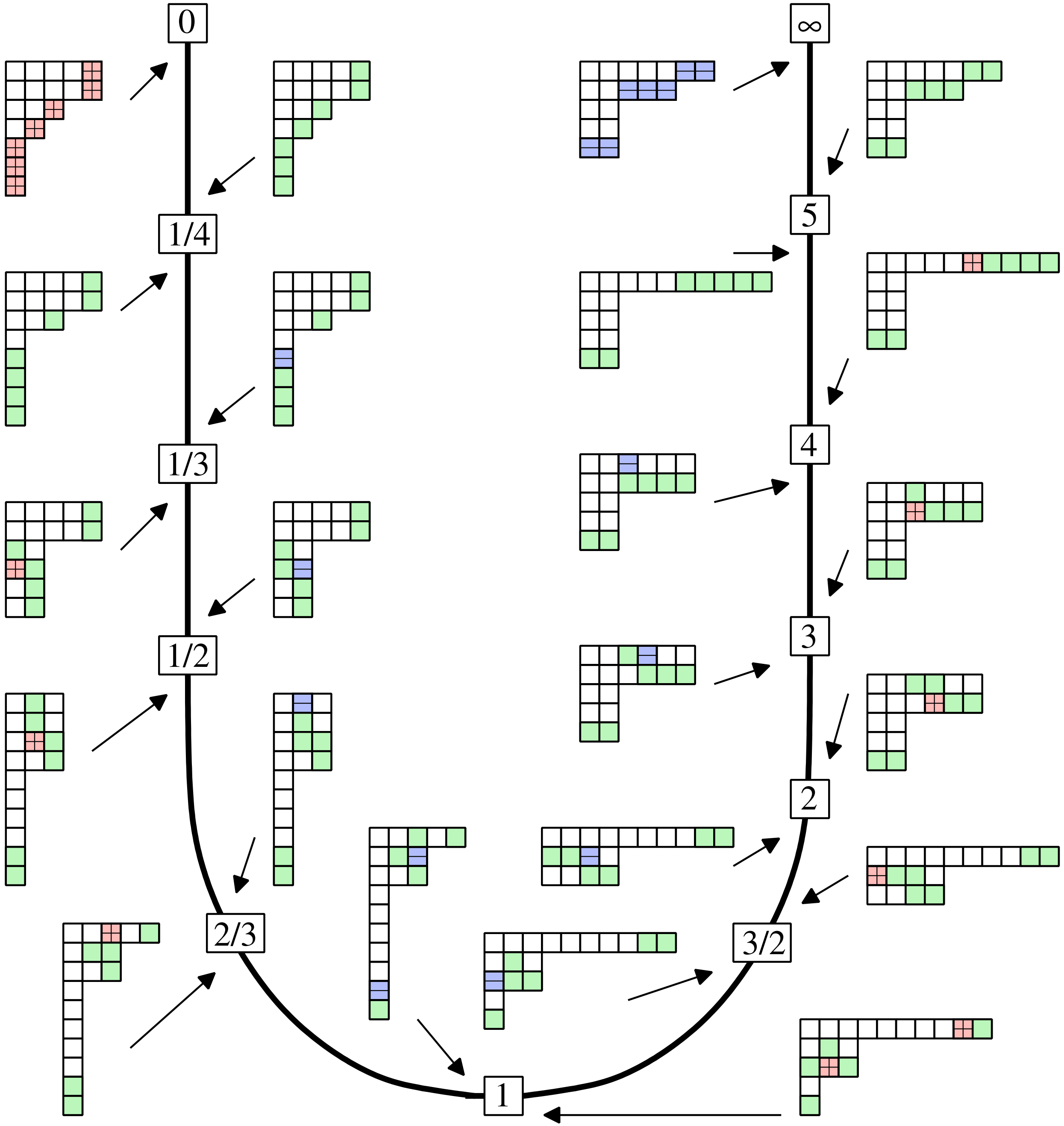
	$c^{-}_{r/s}$	$m_{r/s}$	$c^{+}_{r/s}$	$\lvert\lambda\rvert$
Only on graph		X		X
On Eulerian tour	X		X	

Arrival tree



$c^{\pm}_{r/s} \longrightarrow c^{\mp}_{r/s}$ involution





0
1/4
1/3
1/2
2/3
1
3/2
2
3
4
5
∞

Rephrasing the statistics

$$
\begin{aligned}
\text{area}(M) &= A_{\max}(r,s,n) - \sum_{v \in V_M} \lfloor v/s \rfloor N_{\text{out}}(v,M); \\
\text{mid}(M) &= A_{\max}(r,s,n) - \sum_{v,w \in V_M} E_{\text{in}}(v,M) N_{\text{in}}(w,M) \chi(v \geq w); \\
\mathbf{ctot}(M) &= \sum_{v \in V_M} E_{\text{in}}(v,M) N_{\text{in}}(v,M) - (n - E_{\text{in}}(0,M)).
\end{aligned}
$$

Theorem For any $\mu \in \mathrm{Par}_{r,s,n}$, we have:

$$
c^{+}_{r/s}(\mu) = \sum_{v \in V_M} \mathrm{inv}(w^v(\mu)), \qquad c^{-}_{r/s}(\mu) = \sum_{v \in V_M} \mathrm{inv}(y^v(\mu));
$$

$$
|\mu| = \text{area}(M), \quad \text{mid}_{r/s}(\mu) = \text{mid}(M), \quad \mathbf{ctot}_{r/s}(\mu) = \mathbf{ctot}(M).
$$