Cyclic sieving for longest reduced words in the hyperoctahedral group

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Longest words in the hyperoctahedral group

Hyperoctahedral group: B_n

Generators: $s_0, s_1, \ldots, s_{n-1}$

Relations: $\left\{ \begin{array}{rcl} s_i^2 & = & 1 \\ s_i\, s_j & = & s_j\, s_i \;\text{ for }\; |i-j| \geq 2 \\ s_i\, s_{i+1}\, s_i & = & s_{i+1}\, s_i\, s_{i+1} \;\text{ for }\; i \geq 1 \\ s_0\, s_1\, s_0\, s_1 & = & s_1\, s_0\, s_1\, s_0. \end{array} \right\}$

Longest element: w_0, of length $\ell(w_0) = n^2$

$R(w_0) = \{\text{reduced words for } w_0\}$

Cyclic rotation: $a_1 a_2 \cdots a_{n^2} \stackrel{\omega}{\mapsto} a_2 \cdots a_{n^2} a_1$

Q: What are the sizes of the orbits with respect to this action?

Example: B_3

Orbit of size 9:

$$\xrightarrow{\omega} \mathbf{0}10212012 \xrightarrow{\omega} 10212012\mathbf{0} \xrightarrow{\omega} 021201201 \xrightarrow{\omega} 212012010$$
$$\xrightarrow{\omega} 120120102 \xrightarrow{\omega} 201201021 \xrightarrow{\omega} 012010212 \xrightarrow{\omega} 120102120$$
$$\xrightarrow{\omega} 201021201 \xrightarrow{\omega}$$

Orbit of size 3:

$$\xrightarrow{\omega} \mathbf{0}12012012 \xrightarrow{\omega} 12012012\mathbf{0} \xrightarrow{\omega} 201201201 \xrightarrow{\omega}$$

- 42 words fixed by 0 rotations,
- 6 words fixed by 3 rotations (example: 012012012),
- 6 words fixed by 6 rotations,
- 0 words fixed by any other number of rotations (mod 9),

Square Young tableaux

$SYT(n^n) = \{\text{Standard Young tableaux of shape } n^n\}$

Promotion:

$$\begin{array}{|c|c|c|}\hline 1&2&4\\\hline 3&5&6\\\hline 7&8&9\\\hline\end{array} \longrightarrow \begin{array}{|c|c|c|}\hline &2&4\\\hline 3&5&6\\\hline 7&8&9\\\hline\end{array} \xrightarrow{\text{jdt}} \begin{array}{|c|c|c|}\hline 2&4&6\\\hline 3&5&9\\\hline 7&8&\\\hline\end{array} \xrightarrow{-1} \begin{array}{|c|c|c|}\hline 1&3&5\\\hline 2&4&8\\\hline 6&7&\\\hline\end{array} \longrightarrow \begin{array}{|c|c|c|}\hline 1&3&5\\\hline 2&4&8\\\hline 6&7&9\\\hline\end{array}$$

So

$$\begin{array}{|c|c|c|}\hline 1&2&4\\\hline 3&5&6\\\hline 7&8&9\\\hline\end{array} \xrightarrow{\omega} \begin{array}{|c|c|c|}\hline 1&3&5\\\hline 2&4&8\\\hline 6&7&9\\\hline\end{array}$$

Example: $SYT(3^3)$

Promotion orbit of size 3

$$\xrightarrow{\omega} \begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline 4 & 5 & 6 \\ \hline 7 & 8 & 9 \\ \hline \end{array} \xrightarrow{\omega} \begin{array}{|c|c|c|}\hline 1 & 2 & 5 \\ \hline 3 & 4 & 8 \\ \hline 6 & 7 & 9 \\ \hline \end{array} \xrightarrow{\omega} \begin{array}{|c|c|c|}\hline 1 & 3 & 4 \\ \hline 2 & 6 & 7 \\ \hline 5 & 8 & 9 \\ \hline \end{array} \xrightarrow{\omega}$$

- 42 tableaux fixed by 0 promotions,
- 6 tableaux fixed by 3 promotions,
- 6 tableaux fixed by 6 promotions,
- 0 tableaux fixed by any other number of promotions (mod 9),

Cyclic sieving phenomenon (CSP)

X a set.

$C = \langle \omega \rangle$ a finite cyclic group acting on X.

$X(q) \in \mathbb{Z}(q)$ a polynomial in q.

The triple $(X, C, X(q))$ exhibits CSP if for all $d \geq 0$, the number of elements fixed by ω^d is $X(\zeta^d)$, where ζ is a primitive root of unity of order $|C|$.

Cyclic sieving in $SYT(n^n)$

Theorem (Rhoades)

The following triple exhibits CSP:

$X = SYT(n^n)$

$\omega =$ *promotion*

$$X(q) = \frac{[n^2]!_q}{\prod_{(i,j)\in(n^n)} [h_{i,j}]_q}$$ *(the q-hook polynomial)*

- $X(\zeta^0) = X(1) = 42,$
- $X(\zeta^3) = 6,$
- $X(\zeta^6) = 6,$
- $X(\zeta^i) = 0$ for $i \neq 0, 3,$ or $6 \pmod 9$.

Main theorem

Major index: sum of the positions of the descents

$$w = 0\mathbf{1}0\mathbf{2}1\mathbf{2}012$$

$$\mathrm{maj}(w) = 2 + 4 + 6 = 12$$

Theorem (Petersen - S.)

The following triple exhibits CSP:

$X = R(w_0)$ *(the set of reduced words for w_0)*

$\omega =$ *cyclic rotation*

$$X(q) = q^{-n\binom{n}{2}} \sum_{w \in R(w_0)} q^{\mathrm{maj}(w)}$$

Sketch of proof of the main theorem

- Bijection H between $R(w_0)$ and $SYT(n^n)$.
- H behaves well with respect to CSP.
  - Cyclic rotation corresponds to promotion.
  - Polynomials are the same.
- CSP follows from Rhoades's theorem.
- Note: The bijection goes through an intermediate object: *double staircases*.

Shifted double staircases

$SYT'(2n-1, 2n-3, \ldots, 1) = \{\text{shifted double staircases}\}$
Promotion:

$$\begin{ytableau} 1 & 2 & 4 & 6 & 9 \\ \none & 3 & 5 & 8 \\ \none & \none & 7 \end{ytableau} \longrightarrow \begin{ytableau} \ & 2 & 4 & 6 & 9 \\ \none & 3 & 5 & 8 \\ \none & \none & 7 \end{ytableau} \xrightarrow{\text{jdt}} \begin{ytableau} 2 & 3 & 4 & 6 & 9 \\ \none & 5 & 7 & 8 \\ \none & \none & \ \end{ytableau} \longrightarrow \begin{ytableau} 1 & 2 & 3 & 5 & 8 \\ \none & 4 & 6 & 7 \\ \none & \none & 9 \end{ytableau}$$

So

$$\begin{ytableau} 1 & 2 & 4 & 6 & 9 \\ \none & 3 & 5 & 8 \\ \none & \none & 7 \end{ytableau} \xrightarrow{\omega} \begin{ytableau} 1 & 2 & 3 & 5 & 8 \\ \none & 4 & 6 & 7 \\ \none & \none & 9 \end{ytableau}$$

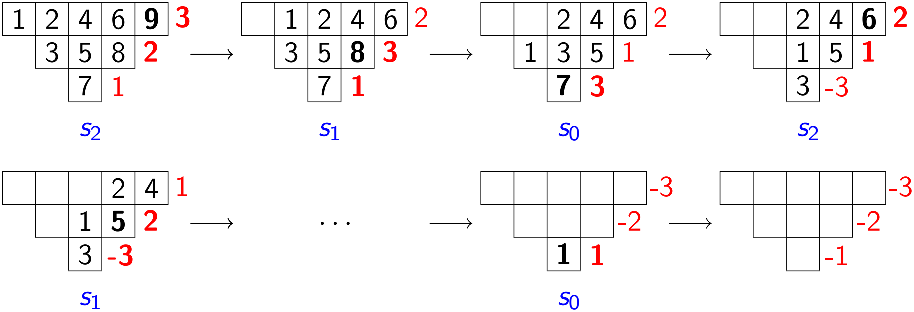
Bijection between longest reduced words and shifted double staircases (Haiman)

$$
\underset{\textcolor{blue}{s_2}}{\begin{array}{|c|c|c|c|c|}\hline 1 & 2 & 4 & 6 & \mathbf{9} \\ \hline \multicolumn{1}{c|}{} & 3 & 5 & 8 & \multicolumn{1}{c}{} \\ \cline{2-4} \multicolumn{2}{c|}{} & 7 & \multicolumn{2}{c}{} \\ \cline{3-3} \end{array}}
\longrightarrow
\underset{\textcolor{blue}{s_1}}{\begin{array}{|c|c|c|c|c|}\hline & 1 & 2 & 4 & 6 \\ \hline \multicolumn{1}{c|}{} & 3 & 5 & \mathbf{8} & \multicolumn{1}{c}{} \\ \cline{2-4} \multicolumn{2}{c|}{} & 7 & \multicolumn{2}{c}{} \\ \cline{3-3} \end{array}}
\longrightarrow
\underset{\textcolor{blue}{s_0}}{\begin{array}{|c|c|c|c|c|}\hline & & 2 & 4 & 6 \\ \hline \multicolumn{1}{c|}{} & 1 & 3 & 5 & \multicolumn{1}{c}{} \\ \cline{2-4} \multicolumn{2}{c|}{} & \mathbf{7} & \multicolumn{2}{c}{} \\ \cline{3-3} \end{array}}
\longrightarrow
\underset{\textcolor{blue}{s_2}}{\begin{array}{|c|c|c|c|c|}\hline & & 2 & 4 & \mathbf{6} \\ \hline \multicolumn{1}{c|}{} & & 1 & 5 & \multicolumn{1}{c}{} \\ \cline{2-4} \multicolumn{2}{c|}{} & 3 & \multicolumn{2}{c}{} \\ \cline{3-3} \end{array}}
$$

$$
\underset{\textcolor{blue}{s_1}}{\begin{array}{|c|c|c|c|c|}\hline & & & 2 & 4 \\ \hline \multicolumn{1}{c|}{} & & 1 & \mathbf{5} & \multicolumn{1}{c}{} \\ \cline{2-4} \multicolumn{2}{c|}{} & 3 & \multicolumn{2}{c}{} \\ \cline{3-3} \end{array}}
\longrightarrow \quad \cdots \quad \longrightarrow
\underset{\textcolor{blue}{s_0}}{\begin{array}{|c|c|c|c|c|}\hline & & & & \\ \hline \multicolumn{1}{c|}{} & & & & \multicolumn{1}{c}{} \\ \cline{2-4} \multicolumn{2}{c|}{} & \mathbf{1} & \multicolumn{2}{c}{} \\ \cline{3-3} \end{array}}
\longrightarrow
\begin{array}{|c|c|c|c|c|}\hline & & & & \\ \hline \multicolumn{1}{c|}{} & & & & \multicolumn{1}{c}{} \\ \cline{2-4} \multicolumn{2}{c|}{} & & \multicolumn{2}{c}{} \\ \cline{3-3} \end{array}
$$

Define $H_1\left(\begin{array}{|c|c|c|c|c|}\hline 1 & 2 & 4 & 6 & 9 \\ \hline \multicolumn{1}{c|}{} & 3 & 5 & 8 & \multicolumn{1}{c}{} \\ \cline{2-4} \multicolumn{2}{c|}{} & 7 & \multicolumn{2}{c}{} \\ \cline{3-3} \end{array}\right) = \textcolor{blue}{s_0 s_1 s_0 s_2 s_1 s_2 s_0 s_1 s_2}$. (or $\textcolor{blue}{010212012}$)

Bijection between longest reduced words and shifted double staircases (Haiman)



Define $H_1\left(\begin{array}{|c|c|c|c|c|} \hline 1 & 2 & 4 & 6 & 9 \\ \hline \multicolumn{1}{c|}{} & 3 & 5 & 8 & \multicolumn{1}{c}{} \\ \cline{2-4} \multicolumn{1}{c}{} & \multicolumn{1}{c|}{} & 7 & \multicolumn{1}{c}{} & \multicolumn{1}{c}{} \\ \cline{3-3} \end{array}\right) = s_0 s_1 s_0 s_2 s_1 s_2 s_0 s_1 s_2$. (or 010212012)

Bijection between shifted double staircases and square Young tableaux

Theorem (Haiman)

The sets $SYT(n^n)$ and $SYT'(2n-1, 2n-3, \ldots, 1)$ are in bijection.

Example

$$H_2\left(\begin{array}{|c|c|c|}\hline 1&2&4\\\hline 3&5&6\\\hline 7&8&9\\\hline\end{array}\right) = \begin{array}{ccccc} 1&2&4&6&9\\ &3&5&8& \\ &&7&& \end{array}.$$

Bijection:

$$\begin{array}{ccccc} \square&\square&1&2&4\\ &\bullet&3&5&6\\ &&7&8&9 \end{array} \xrightarrow{\text{jdt}} \begin{array}{ccccc} \square&\bullet&1&2&4\\ &3&5&6&9\\ &&7&8& \end{array} \xrightarrow{\text{jdt}} \begin{array}{ccccc} \bullet&1&2&4&9\\ &3&5&6& \\ &&7&8& \end{array} \xrightarrow{\text{jdt}} \begin{array}{ccccc} 1&2&4&6&9\\ &3&5&8& \\ &&7&& \end{array}$$

Proof of the main theorem

$$\begin{array}{ccccc} SYT & \xrightarrow{H_2} & SYT' & \xrightarrow{H_1} & R(w_0) \\ \begin{array}{|c|c|c|}\hline 1 & 2 & 4 \\ \hline 3 & 5 & 6 \\ \hline 7 & 8 & 9 \\ \hline \end{array} & \mapsto & \begin{array}{ccccc} \boxed{1} & \boxed{2} & \boxed{4} & \boxed{6} & \boxed{9} \\ & \boxed{3} & \boxed{5} & \boxed{8} & \\ & & \boxed{7} & & \end{array} & \mapsto & 101212012 \end{array}$$

Lemma (Petersen - S.)

$H_1 \circ H_2$ takes promotion in $SYT(n^n)$ to cyclic rotation in $R(w_0)$.

Lemma (Petersen - S.)

The q-hook polynomial in for (n^n) is $q^{-n\binom{n}{2}}$ times the major index generating function in $R(w_0)$.

$$\frac{[n^2!]_q}{\prod_{(i,j)\in(n^n)}[h_{i,j}]_q} = q^{-n\binom{n}{2}} \sum_{w\in R(w_0)} q^{\mathrm{maj}(w)}.$$

Questions

- Is there an explicit CSP for the set of shifted double staircases?
- Are there similar CSP results for longest words in other Coxeter groups?
- Rhoades's Theorem is the type A version of a more general conjecture regarding cominuscule posets. This has been proved for all finite types except B_n and checked [Dilks, Petersen, Stembridge, Yong] for B_n with $n \leq 6$.

Thank you

T. Kyle Petersen and Luis Serrano, *Cyclic sieving for longest reduced words in the hyperoctahedral group*. arXiv: 0905.2650.