Cyclic sieving for longest reduced words in the hyperoctahedral group

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Longest words in the hyperoctahedral group

Hyperoctahedral group: B_n

Generators: $s_0, s_1, \ldots, s_{n-1}$

Relations: $$\left\{ \begin{array}{rcl} s_i^2 & = & 1 \\ s_i\, s_j & = & s_j\, s_i \ \text{ for } \ |i-j| \geq 2 \\ s_i\, s_{i+1}\, s_i & = & s_{i+1}\, s_i\, s_{i+1} \ \text{ for } \ i \geq 1 \\ s_0\, s_1\, s_0\, s_1 & = & s_1\, s_0\, s_1\, s_0. \end{array} \right\}$$

Longest element: w_0, of length $\ell(w_0) = n^2$

$R(w_0) = \{\text{reduced words for } w_0\}$

Cyclic rotation: $a_1 a_2 \cdots a_{n^2} \stackrel{\omega}{\mapsto} a_2 \cdots a_{n^2} a_1$

Q: What are the sizes of the orbits with respect to this action?

Example: B_3

Orbit of size 9:

$$\xrightarrow{\omega} \mathbf{0}10212012 \xrightarrow{\omega} 10212012\mathbf{0} \xrightarrow{\omega} 021201201 \xrightarrow{\omega} 212012010$$

$$\xrightarrow{\omega} 120120102 \xrightarrow{\omega} 201201021 \xrightarrow{\omega} 012010212 \xrightarrow{\omega} 120102120$$

$$\xrightarrow{\omega} 201021201 \xrightarrow{\omega}$$

Orbit of size 3:

$$\xrightarrow{\omega} \mathbf{0}12012012 \xrightarrow{\omega} 12012012\mathbf{0} \xrightarrow{\omega} 201201201 \xrightarrow{\omega}$$

Example: B_3

Orbit of size 9:

$$\xrightarrow{\omega} \mathbf{0}10212012 \xrightarrow{\omega} 10212012\mathbf{0} \xrightarrow{\omega} 021201201 \xrightarrow{\omega} 212012010$$
$$\xrightarrow{\omega} 120120102 \xrightarrow{\omega} 201201021 \xrightarrow{\omega} 012010212 \xrightarrow{\omega} 120102120$$
$$\xrightarrow{\omega} 201021201 \xrightarrow{\omega}$$

Orbit of size 3:

$$\xrightarrow{\omega} \mathbf{0}12012012 \xrightarrow{\omega} 12012012\mathbf{0} \xrightarrow{\omega} 201201201 \xrightarrow{\omega}$$

42 words in $R(w_0)$:

- 42 words fixed by 0 rotations,
- 6 words fixed by 3 rotations (example: 012012012),
- 6 words fixed by 6 rotations,
- 0 words fixed by any other number of rotations (mod 9),

Square Young tableaux

$SYT(n^n) = \{\text{Standard Young tableaux of shape } n^n\}$

Promotion:

$$\begin{array}{|c|c|c|}\hline 1 & 2 & 4 \\ \hline 3 & 5 & 6 \\ \hline 7 & 8 & 9 \\ \hline \end{array} \longrightarrow \begin{array}{|c|c|c|}\hline & 2 & 4 \\ \hline 3 & 5 & 6 \\ \hline 7 & 8 & 9 \\ \hline \end{array} \xrightarrow{\text{jdt}} \begin{array}{|c|c|c|}\hline 2 & 4 & 6 \\ \hline 3 & 5 & 9 \\ \hline 7 & 8 & \\ \hline \end{array} \xrightarrow{-1} \begin{array}{|c|c|c|}\hline 1 & 3 & 5 \\ \hline 2 & 4 & 8 \\ \hline 6 & 7 & \\ \hline \end{array} \longrightarrow \begin{array}{|c|c|c|}\hline 1 & 3 & 5 \\ \hline 2 & 4 & 8 \\ \hline 6 & 7 & 9 \\ \hline \end{array}$$

So

$$\begin{array}{|c|c|c|}\hline 1 & 2 & 4 \\ \hline 3 & 5 & 6 \\ \hline 7 & 8 & 9 \\ \hline \end{array} \xrightarrow{\omega} \begin{array}{|c|c|c|}\hline 1 & 3 & 5 \\ \hline 2 & 4 & 8 \\ \hline 6 & 7 & 9 \\ \hline \end{array}$$

Example: $SYT(3^3)$

Promotion orbit of size 3

$$\xrightarrow{\omega} \begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline 4 & 5 & 6 \\ \hline 7 & 8 & 9 \\ \hline \end{array} \xrightarrow{\omega} \begin{array}{|c|c|c|}\hline 1 & 2 & 5 \\ \hline 3 & 4 & 8 \\ \hline 6 & 7 & 9 \\ \hline \end{array} \xrightarrow{\omega} \begin{array}{|c|c|c|}\hline 1 & 3 & 4 \\ \hline 2 & 6 & 7 \\ \hline 5 & 8 & 9 \\ \hline \end{array} \xrightarrow{\omega}$$

Example: $SYT(3^3)$

Promotion orbit of size 3

$\xrightarrow{\omega}$

1	2	3
4	5	6
7	8	9

$\xrightarrow{\omega}$

1	2	5
3	4	8
6	7	9

$\xrightarrow{\omega}$

1	3	4
2	6	7
5	8	9

$\xrightarrow{\omega}$

42 tableaux in $SYT(3^3)$:

- 42 tableaux fixed by 0 promotions,
- 6 tableaux fixed by 3 promotions,
- 6 tableaux fixed by 6 promotions,
- 0 tableaux fixed by any other number of promotions (mod 9),

Cyclic sieving phenomenon (CSP)

X a set.

$C = \langle \omega \rangle$ a finite cyclic group acting on X.

$X(q) \in \mathbb{Z}(q)$ a polynomial in q.

The triple $(X, C, X(q))$ exhibits CSP if for all $d \geq 0$, the number of elements fixed by ω^d is $X(\zeta^d)$, where ζ is a primitive root of unity of order $|C|$.

Cyclic sieving in $SYT(n^n)$

Theorem (Rhoades)

The following triple exhibits CSP:

$X = SYT(n^n)$

$\omega =$ *promotion*

$$X(q) = \frac{[n^2]!_q}{\prod_{(i,j)\in(n^n)} [h_{i,j}]_q}$$ *(the q-hook polynomial)*

Cyclic sieving in $SYT(n^n)$

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In $SYT(3^3)$, for $\zeta = e^{\frac{2i\pi}{9}}$

- $X(\zeta^0) = X(1) = 42$,
- $X(\zeta^3) = 6$,
- $X(\zeta^6) = 6$,
- $X(\zeta^i) = 0$ for $i \neq 0, 3,$ or $6 \pmod 9$.

Main theorem

Major index: sum of the positions of the descents

$$w = 0\mathbf{1}0\mathbf{2}1\mathbf{2}012$$

$$\mathrm{maj}(w) = 2 + 4 + 6 = 12$$

Theorem (Petersen - S.)

The following triple exhibits CSP:

$X = R(w_0)$ *(the set of reduced words for w_0)*

$\omega =$ *cyclic rotation*

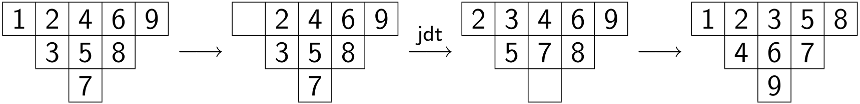
$$X(q) = q^{-n\binom{n}{2}} \sum_{w \in R(w_0)} q^{\mathrm{maj}(w)}$$

Sketch of proof of the main theorem

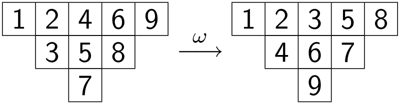
- Bijection H between $R(w_0)$ and $SYT(n^n)$.
- H behaves well with respect to CSP.
  - Cyclic rotation corresponds to promotion.
  - Polynomials are the same.
- CSP follows from Rhoades's theorem.
- Note: The bijection goes through an intermediate object: *double staircases*.

Shifted double staircases

$SYT'(2n-1, 2n-3, \ldots, 1) = \{\text{shifted double staircases}\}$
Promotion:



So



Bijection between longest reduced words and shifted double staircases (Haiman)

1	2	4	6	**9**
	3	5	8	
		7		

s_2

Bijection between longest reduced words and shifted double staircases (Haiman)

1	2	4	6	**9**
	3	5	8	
		7		

s_2

$\longrightarrow$

	1	2	4	6
	3	5	**8**	
		7		

s_1

Bijection between longest reduced words and shifted double staircases (Haiman)

1	2	4	6	**9**
	3	5	8	
		7		

s_2

$\longrightarrow$

	1	2	4	6
	3	5	**8**	
		7		

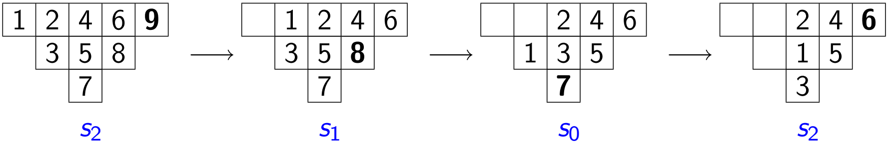
s_1

$\longrightarrow$

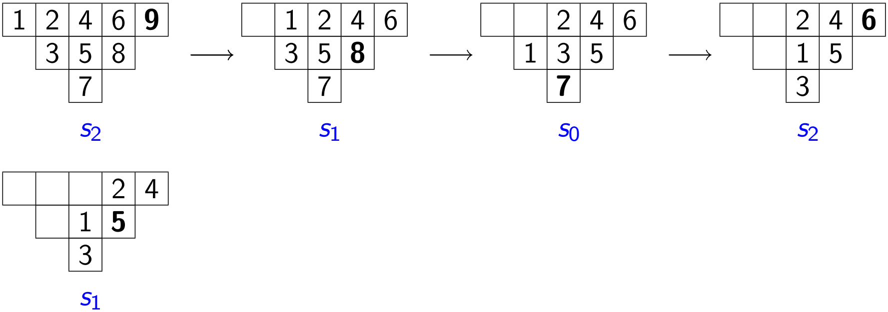
		2	4	6
	1	3	5	
		7		

s_0

Bijection between longest reduced words and shifted double staircases (Haiman)



Bijection between longest reduced words and shifted double staircases (Haiman)



Bijection between longest reduced words and shifted double staircases (Haiman)

1	2	4	6	**9**
	3	5	8	
		7		

s_2

$\longrightarrow$

	1	2	4	6
	3	5	**8**	
		7		

s_1

$\longrightarrow$

		2	4	6
	1	3	5	
		7		

s_0

$\longrightarrow$

		2	4	**6**
		1	5	
		3		

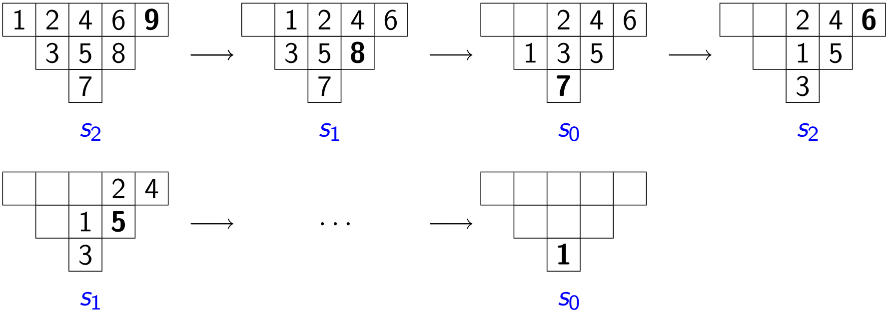
s_2

			2	4
		1	**5**	
		3		

s_1

$\longrightarrow$ $\cdots$

Bijection between longest reduced words and shifted double staircases (Haiman)



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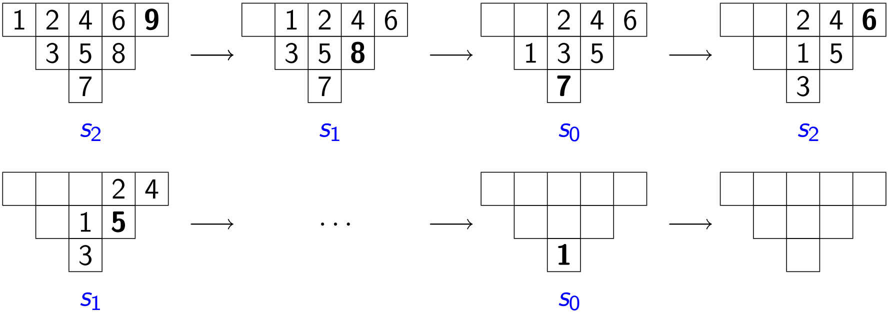
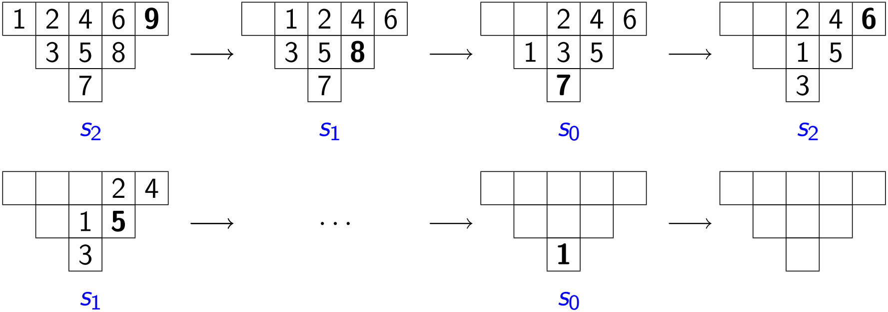


Bijection between longest reduced words and shifted double staircases (Haiman)



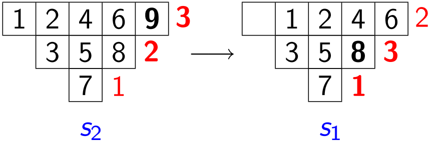
Define $H_1\left(\begin{array}{ccccc}1&2&4&6&9\\ &3&5&8&\\ &&7&&\end{array}\right) = s_0 s_1 s_0 s_2 s_1 s_2 s_0 s_1 s_2$. (or 010212012)

Bijection between longest reduced words and shifted double staircases (Haiman)

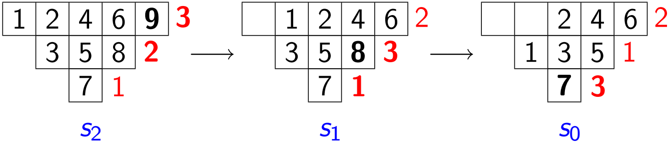
1	2	4	6	**9**	**3**
	3	5	8	**2**	
		7	1		

s_2

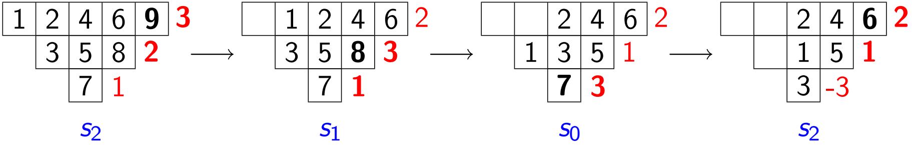
Bijection between longest reduced words and shifted double staircases (Haiman)



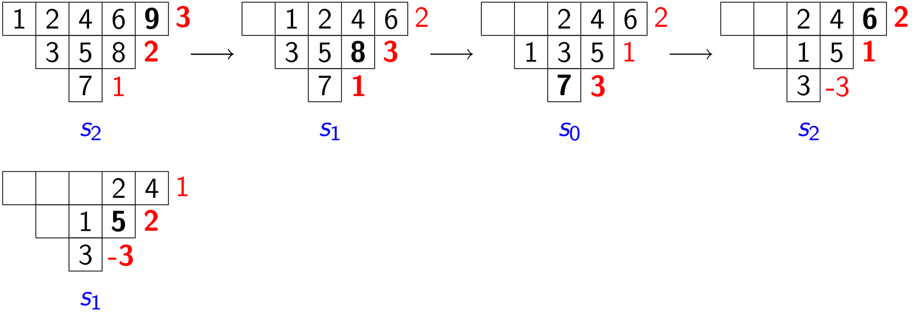
Bijection between longest reduced words and shifted double staircases (Haiman)



Bijection between longest reduced words and shifted double staircases (Haiman)



Bijection between longest reduced words and shifted double staircases (Haiman)



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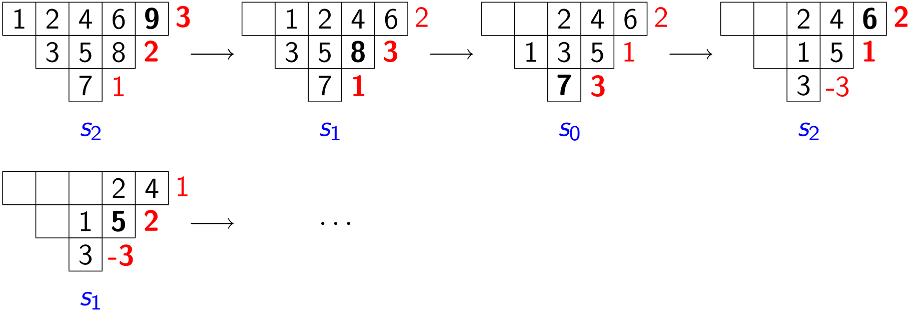

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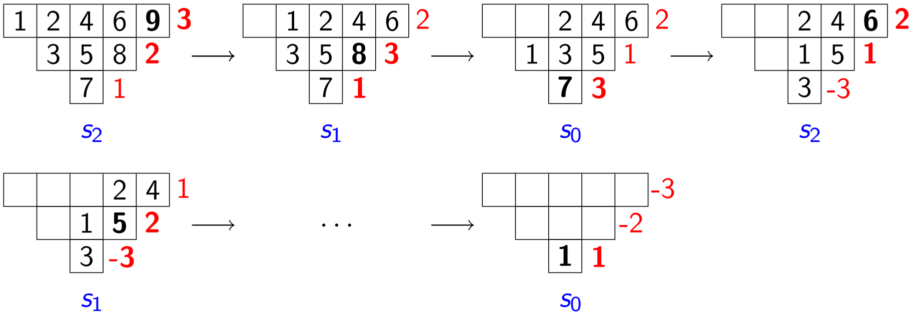

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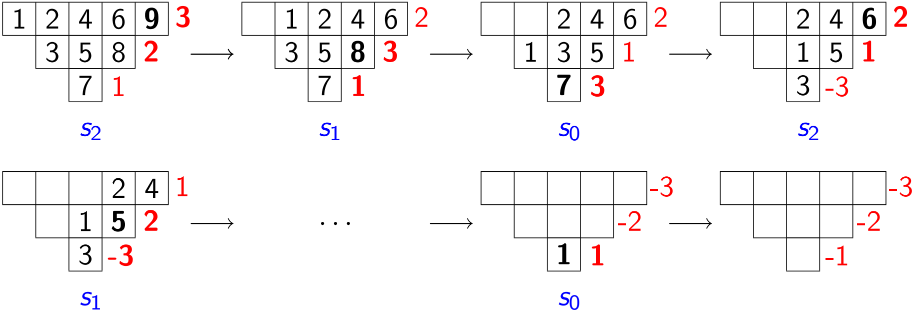

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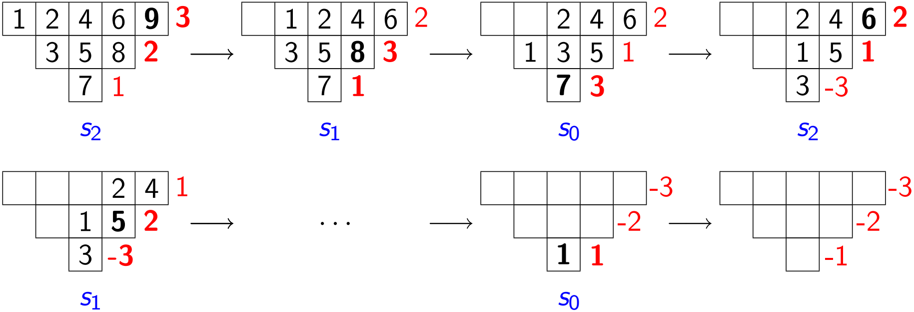

Define $H_1\left(\begin{array}{|c|c|c|c|c|}\hline 1 & 2 & 4 & 6 & 9 \\ \hline \multicolumn{1}{c|}{} & 3 & 5 & 8 & \multicolumn{1}{c}{} \\ \cline{2-4} \multicolumn{2}{c|}{} & 7 & \multicolumn{2}{c}{} \\ \cline{3-3}\end{array}\right) = s_0 s_1 s_0 s_2 s_1 s_2 s_0 s_1 s_2$. (or 010212012)

Bijection between shifted double staircases and square Young tableaux

Theorem (Haiman)

The sets $SYT(n^n)$ and $SYT'(2n-1, 2n-3, \ldots, 1)$ are in bijection.

Example

$$H_2\left(\begin{array}{|c|c|c|}\hline 1 & 2 & 4 \\ \hline 3 & 5 & 6 \\ \hline 7 & 8 & 9 \\ \hline \end{array}\right) = \begin{array}{ccccc} \boxed{1} & \boxed{2} & \boxed{4} & \boxed{6} & \boxed{9} \\ & \boxed{3} & \boxed{5} & \boxed{8} & \\ & & \boxed{7} & & \end{array}.$$

Bijection:

$$\begin{array}{ccccc} \boxed{} & \boxed{} & \boxed{1} & \boxed{2} & \boxed{4} \\ & \boxed{\bullet} & \boxed{3} & \boxed{5} & \boxed{6} \\ & & \boxed{7} & \boxed{8} & \boxed{9} \end{array} \xrightarrow{\text{jdt}} \begin{array}{ccccc} \boxed{} & \boxed{\bullet} & \boxed{1} & \boxed{2} & \boxed{4} \\ & \boxed{3} & \boxed{5} & \boxed{6} & \boxed{9} \\ & & \boxed{7} & \boxed{8} & \end{array} \xrightarrow{\text{jdt}} \begin{array}{ccccc} \boxed{\bullet} & \boxed{1} & \boxed{2} & \boxed{4} & \boxed{9} \\ & \boxed{3} & \boxed{5} & \boxed{6} & \\ & & \boxed{7} & \boxed{8} & \end{array} \xrightarrow{\text{jdt}} \begin{array}{ccccc} \boxed{1} & \boxed{2} & \boxed{4} & \boxed{6} & \boxed{9} \\ & \boxed{3} & \boxed{5} & \boxed{8} & \\ & & \boxed{7} & & \end{array}$$

Proof of the main theorem

$$SYT \xrightarrow{H_2} SYT' \xrightarrow{H_1} R(w_0)$$

1	2	4
3	5	6
7	8	9

$\mapsto$

1	2	4	6	9
	3	5	8	
		7		

$\mapsto$ 101212012

Lemma (Petersen - S.)

$H_1 \circ H_2$ takes promotion in $SYT(n^n)$ to cyclic rotation in $R(w_0)$.

Lemma (Petersen - S.)

The q-hook polynomial in for (n^n) is $q^{-n\binom{n}{2}}$ times the major index generating function in $R(w_0)$.

$$\frac{[n^2!]_q}{\prod_{(i,j)\in(n^n)}[h_{i,j}]_q} = q^{-n\binom{n}{2}} \sum_{w\in R(w_0)} q^{\mathrm{maj}(w)}.$$

Questions

- Is there an explicit CSP for the set of shifted double staircases?
- Are there similar CSP results for longest words in other Coxeter groups?
- Rhoades's Theorem is the type A version of a more general conjecture regarding cominuscule posets. This has been proved for all finite types except B_n and checked [Dilks, Petersen, Stembridge, Yong] for B_n with $n \leq 6$.

Thank you

T. Kyle Petersen and Luis Serrano, *Cyclic sieving for longest reduced words in the hyperoctahedral group*. arXiv: 0905.2650.