Probabilistic Proofs of Hooklength Formulas

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A hooklength formula involving trees

A probabilistic proof

Generalizations and open questions

Outline

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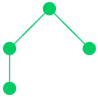
$T =$ 

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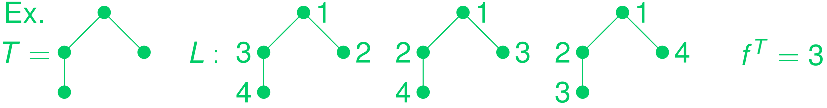

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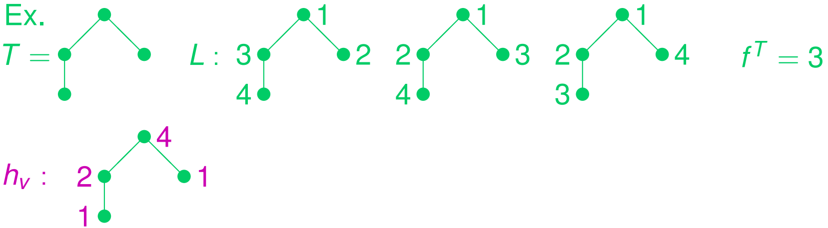

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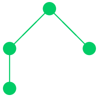

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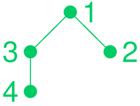

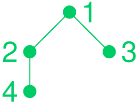

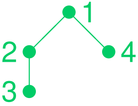

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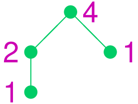

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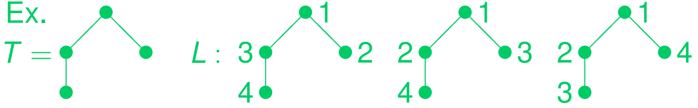

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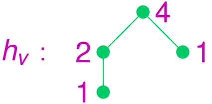

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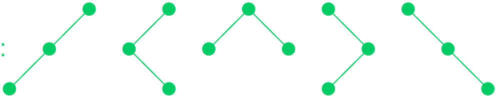

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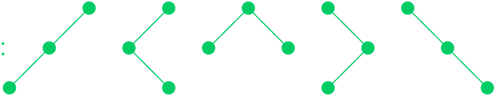

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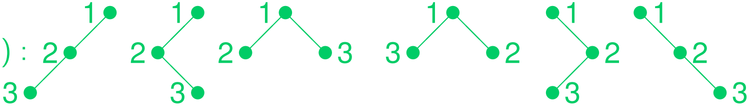

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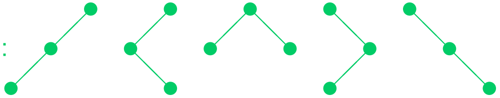

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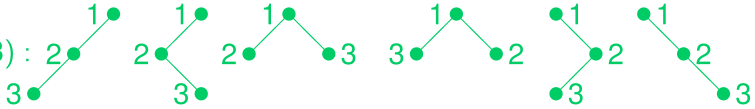

Theorem (Han, 2008)

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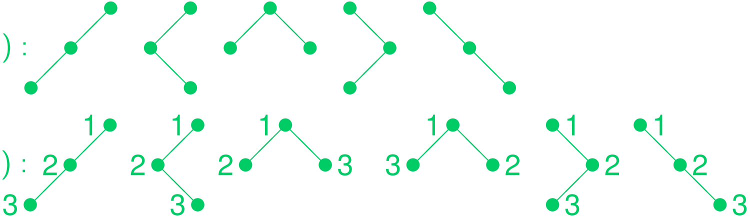

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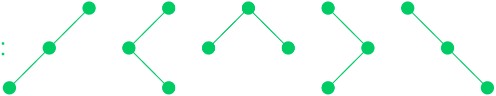

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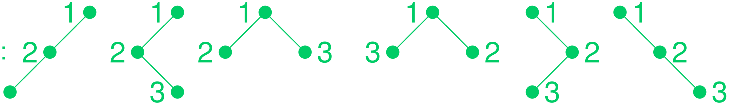

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2. Han's proof is algebraic. Our proof is probabilistic.

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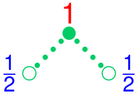

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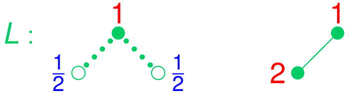

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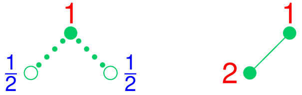

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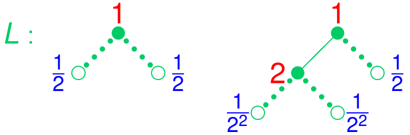

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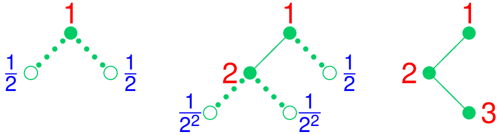

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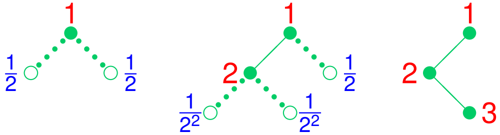

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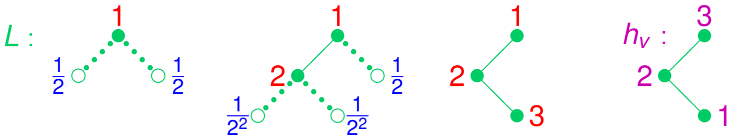

prob $L = 1 \quad \cdot \quad \frac{1}{2} \quad \cdot \quad \frac{1}{2^2}$

(I) prob $L = \prod_{v \in T} 1/2^{h_v - 1}$ if L labels T, and

(II) $\sum_{L \in \mathcal{L}(n)} \text{prob } L = 1$.

For $w \in T$, consider the *depth* of w:

$$d_w = \text{ length of the unique root-to-}w\text{ path.}$$

Algorithm. (a) Let L consist of a root labeled 1.
(b) While $|L| < n$, pick a leaf w to be added to L with label $|L| + 1$ and prob $w = 1/2^{d_w}$.
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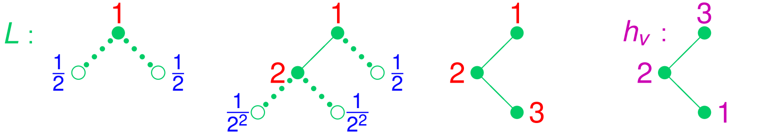

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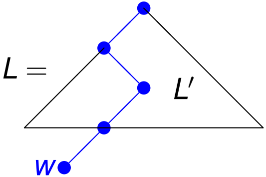

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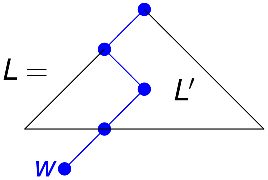

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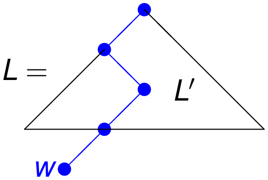

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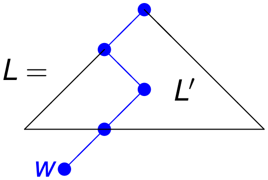

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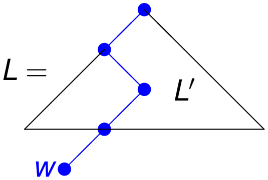

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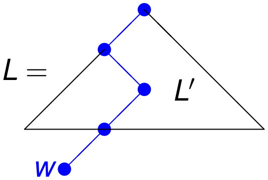

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Outline

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So $\mathrm{wt}(T)$ becomes the number of ways to make T binary and Yang's result implies Han's.

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For any n,

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(e) What is the analogue for tableaux of Han's formulas?