Positivity results for cluster algebras from surfaces

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(Joint work with Ralf Schiffler (University of Connecticut) and Lauren Williams (University of California, Berkeley))

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http//math.mit.edu/∼musiker/ClusterSurfaceAMS.pdf

Outline

1. Introduction: the Laurent phenomenon, and the positivity conjecture of Fomin-Zelevinsky.
2. Fomin-Shapiro-Thurston's theory of cluster algebras arising from triangulated surfaces.
3. Graph theoretic construction for surfaces with or without punctures (joint work with Schiffler and Williams).
4. Examples of this construction.

Introduction to Cluster Algebras

In the late 1990's: Fomin and Zelevinsky were studying total positivity and canonical bases of algebraic groups. They noticed recurring combinatorial and algebraic structures.

Led them to define cluster algebras, which have now been linked to quiver representations, Poisson geometry, Teichmüller theory, tropical geometry, Lie groups, and other topics.

Cluster algebras are a certain class of commutative rings which have a distinguished set of generators that are grouped into overlapping subsets, called clusters, each having the same cardinality.

What is a Cluster Algebra?

Definition (Sergey Fomin and Andrei Zelevinsky 2001) A cluster algebra $\mathcal{A}$ (of geometric type) is a subalgebra of $k(x_1, \ldots, x_n, x_{n+1}, \ldots, x_{n+m})$ constructed cluster by cluster by certain exchange relations.

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The set of all such generators are known as Cluster Variables, and the initial pattern B of exchange relations determines the Seed.

Relations:

Induced by the Binomial Exchange Relations.

A priori, get a tree of exchanges.

In practice, often get identifications among seeds.

In extreme cases, get only a finite number of exchange patterns as tree closes up on itself. Such cluster algebras called finite mutation type.

Sometimes only a finite number of *clusters*. Called finite type.

$$\text{Finite type} \implies \text{Finite mutation type.}$$

Theorem. (FZ 2002) Finite type cluster algebras can be described via the Cartan-Killing classification of Lie algebras.

Cluster Expansions and the Laurent Phenomenon

Example. Let $\mathcal{A}$ be the cluster algebra defined by the initial cluster $\{x_1, x_2, x_3, y_1, y_2, y_3\}$ and the initial exchange pattern

$$x_1x_1' = y_1 + x_2, \quad x_2x_2' = x_1x_3y_2 + 1, \quad x_3x_3' = y_3 + x_2. \quad \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

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$\mathcal{A}$ is of finite type, type A_3 and corresponds to a triangulated hexagon.

$$\left\{ x_1, x_2, x_3, \frac{y_1 + x_2}{x_1}, \frac{x_1 x_3 y_2 + 1}{x_2}, \frac{y_3 + x_2}{x_3}, \frac{x_1 x_3 y_1 y_2 + y_1 + x_2}{x_1 x_2}, \right.$$
$$\left. \frac{x_1 x_3 y_2 y_3 + y_3 + x_2}{x_2 x_3}, \frac{x_1 x_3 y_1 y_2 y_3 + y_1 y_3 + x_2 y_3 + x_2 y_1 + x_2^2}{x_1 x_2 x_3} \right\}.$$

The y_i's are known as principal coefficients.

Theorem. (The Laurent Phenomenon FZ 2001) For any cluster algebra defined by initial seed $(\{x_1, x_2, \ldots, x_{n+m}\}, B)$, all cluster variables of $\mathcal{A}(B)$ are Laurent polynomials in $\{x_1, x_2, \ldots, x_{n+m}\}$

(with no coefficient $x_{n+1}, \ldots, x_{n+m}$ in the denominator).

Because of the Laurent Phenomenon, any cluster variable x_α can be expressed as $\frac{P_\alpha(x_1,\ldots,x_{n+m})}{x_1^{\alpha_1}\cdots x_n^{\alpha_n}}$ where $P_\alpha \in \mathbb{Z}[x_1, \ldots, x_{n+m}]$ and the α_i's $\in \mathbb{Z}$.

The Positivity Conjecture of Fomin and Zelevinsky

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Conjecture. (FZ 2001) For any cluster variable x_α and any initial seed (i.e. initial cluster $\{x_1, \dots, x_{n+m}\}$ and initial exchange pattern B), the polynomial $P_\alpha(x_1, \dots, x_n)$ has nonnegative integer coefficients.

Some Prior Work on Positivity Conjecture

Work of [Carroll-Price 2002] gave expansion formulas for case of Ptolemy algebras, cluster algebras of type A_n with boundary coefficients $(Gr_{2,n+3})$.

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Positivity for arbitrary surfaces [M-Schiffler-Williams 2009].

Main Theorem

Theorem. (Positivity for cluster algebras from surfaces MSW 2009)
Let $\mathcal{A}$ be any cluster algebra arising from a surface (with or without punctures), where the coefficient system is of geometric type, and let Σ be any initial seed.

Then the Laurent expansion of every cluster variable with respect to the seed Σ has non-negative coefficients.

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Due to work of Felikson-Shapiro-Tumarkin, we get

Corollary. Positivity for any seed, for all but 11 skew-symmetric cluster algebras of finite mutation type. (Rank two skew-symmetric cases by Caldero-Reineke)

Cluster Algebras of Triangulated Surfaces

We follow (Fomin-Shapiro-Thurston), based on earlier work of Fock-Goncharov and Gekhtman-Shapiro-Vainshtein.

We have a surface S with a set of marked points M. (If $P \in M$ is in the interior of S, i.e. $S \setminus \delta S$, then P is known an a puncture).

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An arc γ satisfies (we care about arcs up to isotopy)

1. The endpoints of γ are in M.
2. γ does not cross itself.
3. except for the endpoints, γ is disjoint from M and the boundary of S.
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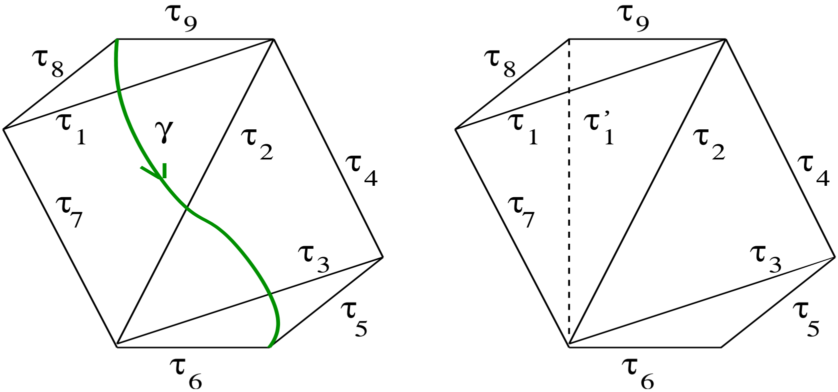
$$\text{Seed} \leftrightarrow \text{Triangulation } T = \{\tau_1, \tau_2, \ldots, \tau_n\}$$

$$\text{Cluster Variable} \leftrightarrow \text{Arc } \gamma \ (x_i \leftrightarrow \tau_i \in T)$$

$$\text{Cluster Mutation} \leftrightarrow \text{Ptolemy Exchanges (Flipping Diagonals).}$$

Example of Hexagon

Consider the triangulated hexagon (S, M) with triangulation T_H.



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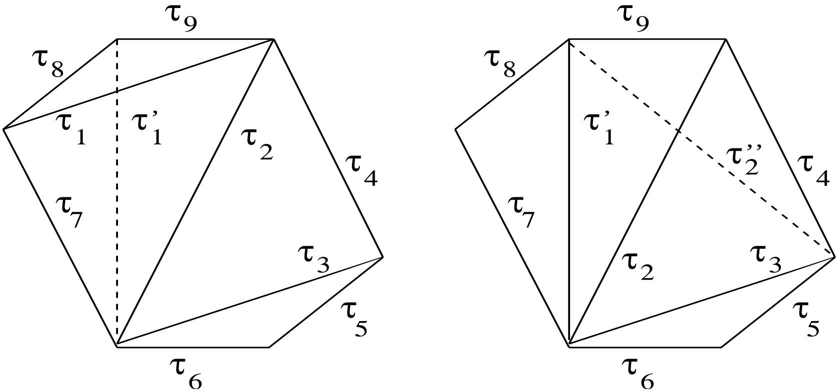

$$\begin{aligned} x_1 x_1' &= \color{red}{y_1}(x_7 x_9) + x_2(x_8) \\ x_2 x_2'' &= \color{red}{y_1 y_2} x_3(x_9) + x_1'(x_4) \end{aligned}$$

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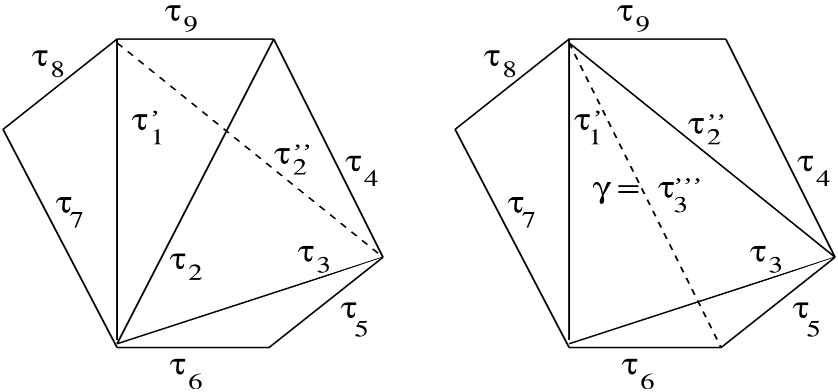

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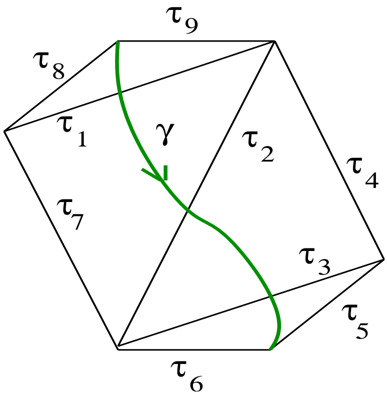
\begin{aligned}
x_1 x_1' &= y_1(x_7 x_9) + x_2(x_8) \\
x_2 x_2'' &= y_1 y_2 x_3(x_9) + x_1'(x_4) \\
x_3 x_3''' &= y_3 x_2''(x_6) + x_1'(x_5)
\end{aligned}
$$

Example of Hexagon (continued)

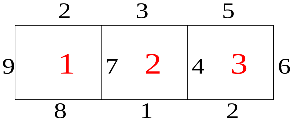


By using the Ptolemy relations on τ_1, τ_2, then τ_3, we obtain

$$x_3''' = x_\gamma = \frac{1}{x_1x_2x_3}\Big(x_2^2(x_5x_8) + y_1x_2(x_5x_7x_9) + y_3x_2(x_4x_6x_8) + y_1y_3(x_4x_6x_7x_9) + y_1y_2y_3x_1x_3(x_6x_9)\Big).$$

Example of Hexagon (continued)

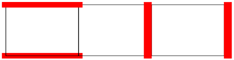
Consider the graph $G_{T_H,\gamma} =$



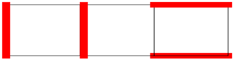
$G_{T_H,\gamma}$ has five perfect matchings $(x_4, x_5, \ldots, x_9 = 1)$:

$(x_9)x_1x_3(x_6)$,

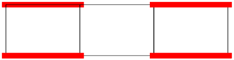
$(x_9x_7x_4x_6)$,



$x_2(x_8)(x_4x_6)$,



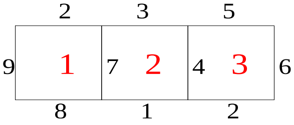
$(x_9x_7)x_2(x_5)$,



$x_2(x_8)x_2(x_5)$.

A perfect matching $M \subseteq E$ is a set of distinguished edges so that every vertex of V is covered exactly once. The weight of a matching M is the product of the weights of the constituent edges, i.e. $x(M) = \prod_{e \in M} x(e)$.

Consider the graph $G_{T_H,\gamma} =$



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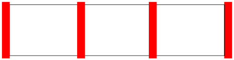

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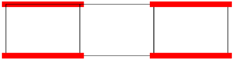

$x_2(x_8)x_2(x_5)$.

$$\frac{x_1x_3y_1y_2y_3 + y_1y_3 + x_2y_3 + x_2y_1 + x_2^2}{x_1x_2x_3}$$

These five monomials exactly match those appearing in the numerator of the expansion of x_γ. The denominator of $x_1x_2x_3$ corresponds to the labels of the three tiles.

A Graph Theoretic Approach

For every triangulation T (in a surface with or without punctures) and an ordinary arc γ through ordinary triangles, we construct a snake graph $G_{T,\gamma}$ such that

$$x_\gamma = \frac{\sum_{\text{perfect matching } M \text{ of } G_{T,\gamma}} x(M)y(M)}{x_1^{e_1(T,\gamma)} x_2^{e_2(T,\gamma)} \cdots x_n^{e_n(T,\gamma)}}.$$

x_γ is cluster variable (corresp. to γ w.r.t. seed given by T) with principal coefficients.

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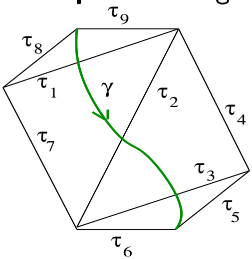
x_γ is cluster variable (corresp. to γ w.r.t. seed given by T) with principal coefficients.

> $e_i(T,\gamma)$ is the crossing number of τ_i and γ (min. int. number),
>
> $x(M)$ is the weight of M,
>
> $y(M)$ is the height of M (to be defined later),

Similar formula will hold for non-ordinary arcs (or through self-folded triangles).

Example 1. Using the above construction for



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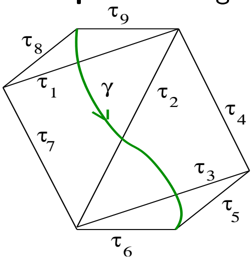

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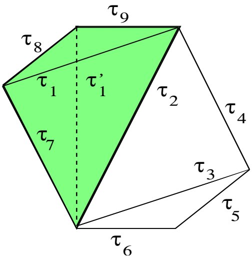

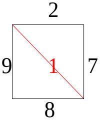

Examples of $G_{T,\gamma}$

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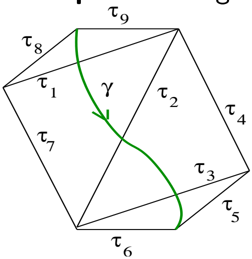

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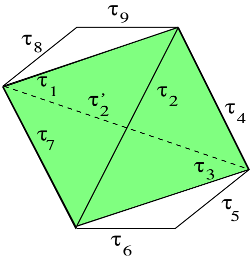

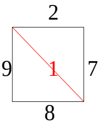

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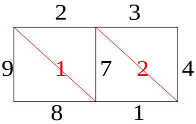

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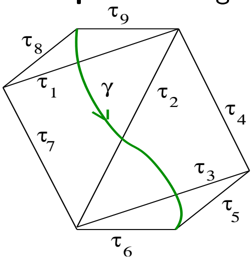

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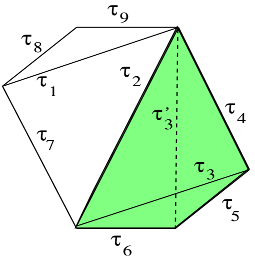

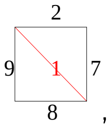

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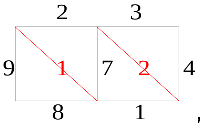

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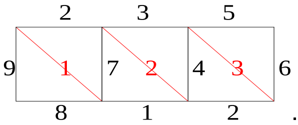

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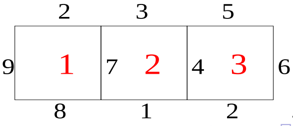
$G_{T_H,\gamma} =$



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Height Functions (of Perfect Matchings of *Snake* Graphs)

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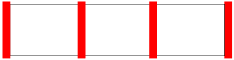
We use height functions which are due to William Thurston, and Conway-Lagarias.

Involves measuring contrast between a given perfect matching M and a fixed minimal matching M_-.

Height Function Examples

Recall that $G_{T_H,\gamma}$ has three faces, labeled 1, 2 and 3. $G_{T_H,\gamma}$ has five perfect matchings $(x_4, x_5, \ldots, x_9 = 1)$:

$y_1 y_2 y_3$,



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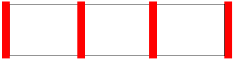

$y_1 y_3$,

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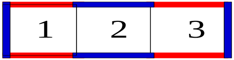
For example, we get heights $y_1 y_2 y_3$, $y_1 y_3$, and y_3 because of superpositions:



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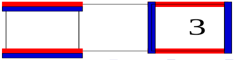
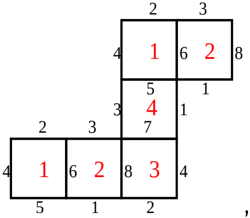
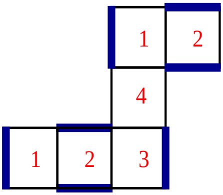
, and



Height Function Examples (continued)

For $G_{T_{A},\gamma} =$ 

, M_- is 

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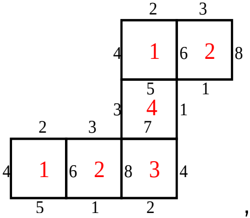
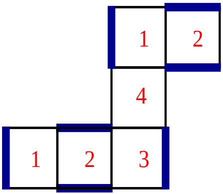
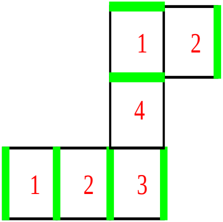
For $G_{T_A,\gamma} =$ 

, M_- is 

. One of the 17 matchings, M, is 

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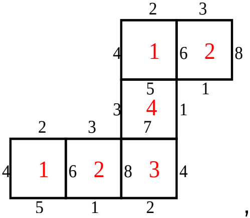
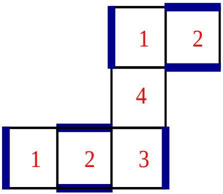
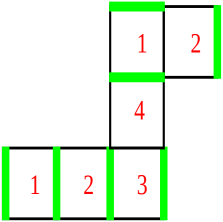
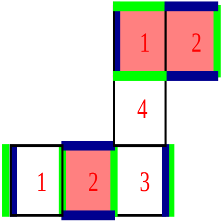
Height Function Examples (continued)

For $G_{T_{A},\gamma} =$ 

, M_- is 

. One of the 17 matchings, M, is 

, so $M \ominus M_- =$ 

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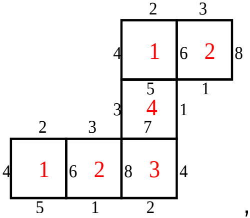
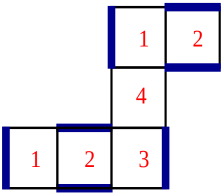
Height Function Examples (continued)

For $G_{T_{A,\gamma}} =$ 

, M_{-} is 

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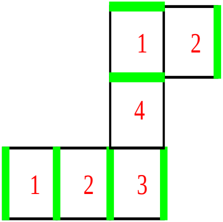
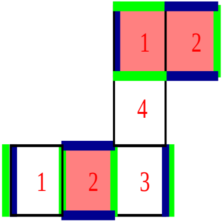
17 matchings, M, is 

, so $M \ominus M_{-} =$ 

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which has height $y_1 y_2^2$. So one of the 17 terms in the cluster expansion of x_γ is (using FST convention) $\frac{x_4(x_6 x_8)x_4(x_5)x_2(x_8)}{x_1^2 x_2^2 x_3 x_4}(y_1 y_2^2)$.

Summary

Theorem. (M-Schiffler-Williams 2009) For every triangulation T of a surface (with or without punctures) and an ordinary arc γ, we construct a snake graph $G_{\gamma,T}$ such that

$$x_\gamma = \frac{\sum_{\text{perfect matching } M \text{ of } G_{\gamma,T}} x(M)y(M)}{x_1^{e_1(T,\gamma)} x_2^{e_2(T,\gamma)} \cdots x_n^{e_n(T,\gamma)}}.$$

Here $e_i(T,\gamma)$ is the crossing number of τ_i and γ, $x(M)$ is the edge-weight of perfect matching M, and $y(M)$ is the height of perfect matching M. (x_γ is cluster variable with principal coefficients.)

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Theorem. (M-Schiffler-Williams 2009) An analogous expansion formula holds for arcs with notches (only arise in the case of a punctured surface).

Corollary. The F-polynomial equals $\sum_M y(M)$, is positive, and has constant term 1.

The g-vector satisfies $\mathbf{x}^g = x(M_-)$.

Example 2 (Ordinary Arc through Self-folded Triangle)

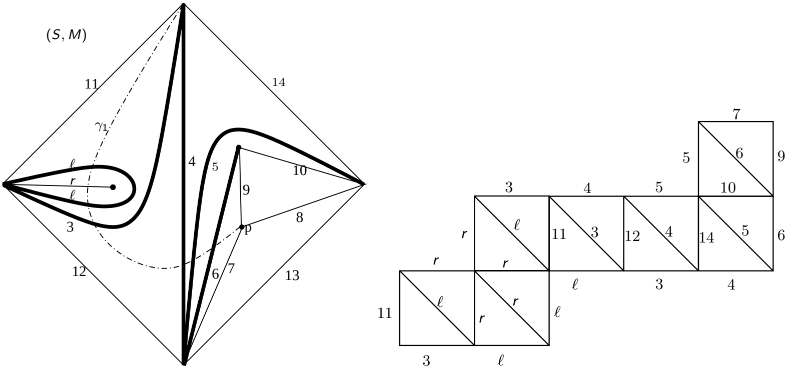

Figure: Ideal Triangulation T° of (S, M) and corresponding Snake Graph $\overline{G}_{T^\circ,\gamma_1}$.

Note the three consecutive tiles of our snake graph with labels ℓ, r and ℓ, as γ_1 traverses the loop ℓ twice and the enclosed radius r.

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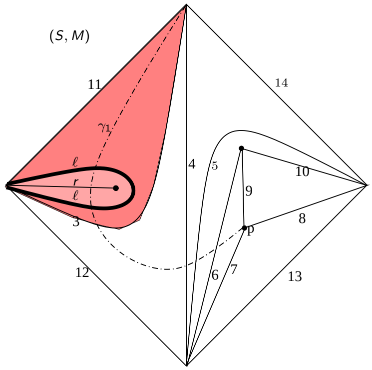

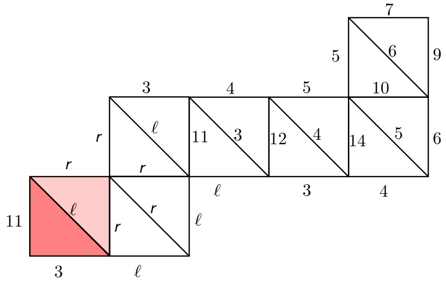

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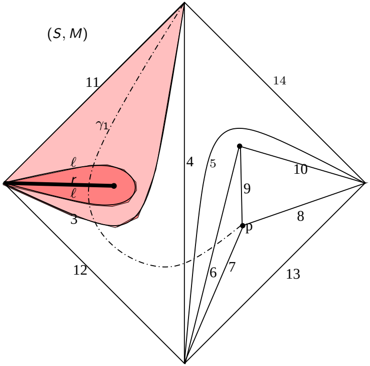

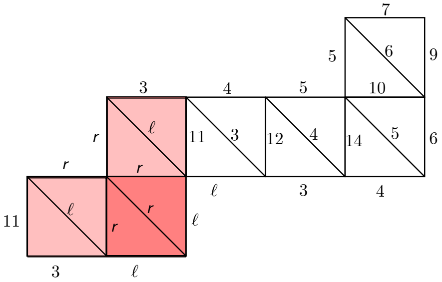

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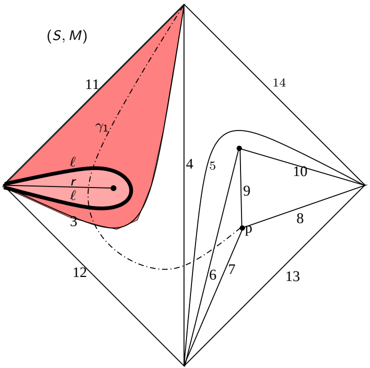

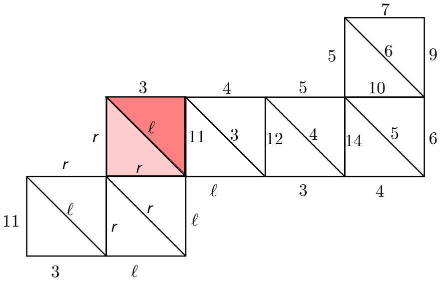

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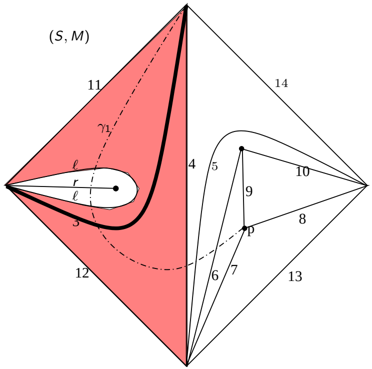

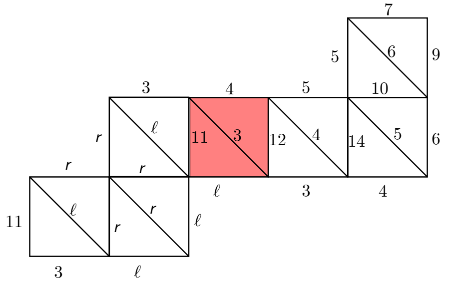

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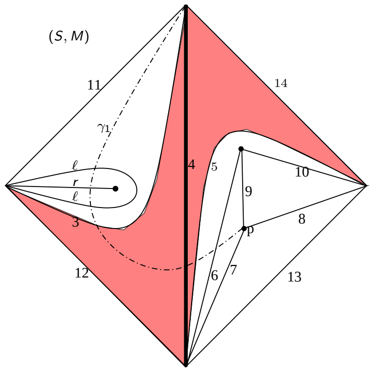

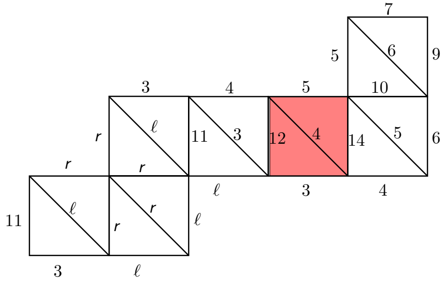

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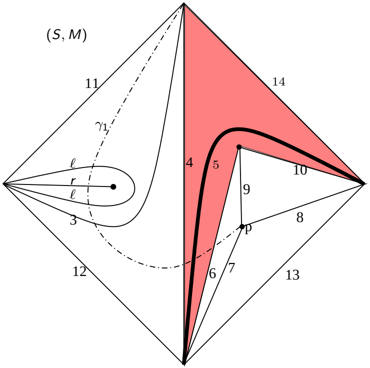

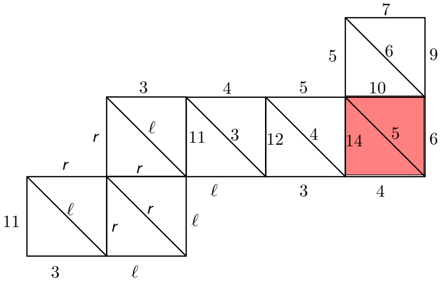

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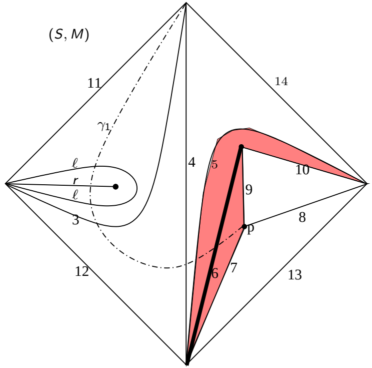

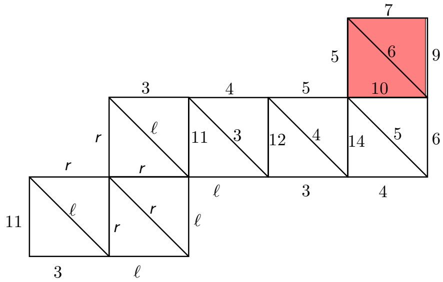

Example 3 (Notched Arc in Punctured Surface)

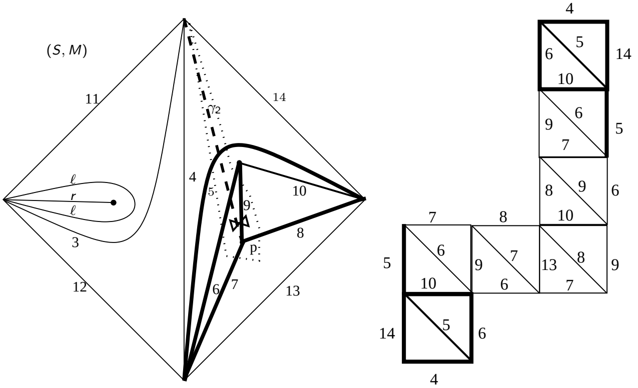

Figure: Ideal Triangulation T° of (S, M) and corresponding Snake Graph $\overline{G}_{T^\circ,\gamma_2}$.

We obtain the Laurent expansion for x_{γ_2} by summing over so called γ-symmetric matchings of G_{T°,γ_2}, those that agree on the two bold ends.

Thank You For Listening

Positivity for Cluster Algebras from Surfaces (with Ralf Schiffler and Lauren Williams), `arXiv:math.CO/0906.0748`

Cluster Expansion Formulas and Perfect Matchings (with Ralf Schiffler), `arXiv:math.CO/0810.3638`

A Graph Theoretic Expansion Formula for Cluster Algebras of Classical Type, `http://www-math.mit.edu/~musiker/Finite.pdf`
(To appear in the Annals of Combinatorics)

Fomin, Shapiro, and Thurston. *Cluster Algebras and Triangulated Surfaces I: Cluster Complexes*, Acta Math. 201 (2008), no. 1, 83–146.

Fomin and Zelevinsky. *Cluster Algebras IV: Coefficients*, Compos. Math. 143 (2007), no. 1, 112–164.

Slides Available at `http//math.mit.edu/~musiker/ClusterSurfaceAMS.pdf`