# A Combinatorial Interpretation of Coefficients Arising in the Quantum Polynomial Ring

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# Quantum Polynomial Ring

**Definition**

$$\mathcal{A}(n;q) = \mathbb{C}[q^{\frac{1}{2}}, q^{-\frac{1}{2}}]\langle x_{1,1}, \ldots, x_{n,n}\rangle$$

such that the following relations hold

$$\begin{aligned} x_{i,\ell}x_{i,k} &= q^{\frac{1}{2}} x_{i,k}x_{i,\ell} \\ x_{j,k}x_{i,k} &= q^{\frac{1}{2}} x_{i,k}x_{j,k} \\ x_{j,k}x_{i,\ell} &= x_{i,\ell}x_{j,k} \\ x_{j,\ell}x_{i,k} &= x_{i,k}x_{j,\ell} + (q^{\frac{1}{2}} - q^{-\frac{1}{2}})x_{i,\ell}x_{j,k}, \end{aligned}$$

for all indices $1 \leq i < j \leq n$ and $1 \leq k < \ell \leq n$.

# Grading

**FACT**

The quantum polynomial ring has a Grading by degree

$$\mathcal{A}(n;q) = \bigoplus_{r\geq 0} \mathcal{A}_r(n;q),$$

where $\mathcal{A}_r(n;q)$ consists of all degree $r$ polynomials.

# Multigrading

## FACT

There is a finer *multigrading* by pairs of multisets

$$\mathcal{A}(n;q) = \bigoplus_{r \geq 0} \bigoplus_{\substack{L,M \\ |L|=|M|=r}} \mathcal{A}_{L,M}(n;q).$$

## Example

$x_{1,1}^3 \in \mathcal{A}_{111,111}(n;q)$,
$x_{1,1}x_{1,2}x_{2,3}$ and $x_{1,1}x_{1,3}x_{2,2} \in \mathcal{A}_{112,123}(n;q)$,
while all three belong to $\mathcal{A}_3(n;q)$.

# Immanant Space

Let $[n] = \{1, \ldots, n\}$.

**Definition**

The *Immanant Space* is defined as

$$\mathcal{A}_{[n],[n]}(n; q) = \mathrm{span}\{x_{1,v_1} \cdots x_{n,v_n} \mid v \in \mathfrak{S}_n\}$$
$$= \mathrm{span}\{x^{e,v} \mid v \in \mathfrak{S}_n\}.$$

We call the set of monomials $\{x^{e,v} \mid v \in \mathfrak{S}_n\}$ the *natural basis* of the immanant space.

# More on the Immanant Space

## FACT

For any fixed $u \in \mathfrak{S}_n$,

$$\mathcal{A}_{[n],[n]}(n;q) = \mathrm{span}\{x_{u_1,v_1} \cdots x_{u_n,v_n} \mid v \in \mathfrak{S}_n\}$$
$$= \mathrm{span}\{x^{u,v} \mid v \in \mathfrak{S}_n\}.$$

## Example

For $n = 3$,

$$\mathcal{A}_{123,123}(3;q) = \mathrm{span}\{x_{3,1}x_{2,2}x_{1,3}, x_{3,2}x_{2,1}x_{1,3}, x_{3,1}x_{2,3}x_{1,2},$$
$$x_{3,2}x_{2,3}x_{1,1}, x_{3,3}x_{2,1}x_{1,2}, x_{3,3}x_{2,2}x_{1,1}\}$$
$$= \mathrm{span}\{x^{w_0,v} \mid v \in \mathfrak{S}_n\}.$$

# Importance of Immanant Space

In some sense, all multigraded components $\mathcal{A}_{L,M}(n;q)$ can be understood in terms of the immanant space $\mathcal{A}_{[n],[n]}(n;q)$.

**Question**

What is the transition matrix relating $\{x^{u,v} \mid v \in \mathfrak{S}_n\}$ for fixed $u$ and $\{x^{e,v} \mid v \in \mathfrak{S}_n\}$?

# $p$-polynomials

**Definition**

Define the polynomials $p_{u,v,w}(q)$ by

$$x^{u,v} = \sum_{w \geq u^{-1}v} p_{u,v,w}(q^{\frac{1}{2}} - q^{\frac{-1}{2}}) x^{e,w}.$$

It turns out $p_{u,v,w}(q) \in \mathbb{N}[q]$, which we can see by looking at the relations.

**Example**

$$\begin{aligned} x^{213,213} &= x_{2,2}x_{1,1}x_{3,3} = x_{1,1}x_{2,2}x_{3,3} + (q^{\frac{1}{2}} - q^{\frac{-1}{2}})x_{1,2}x_{2,1}x_{3,3} \\ &= x^{123,123} + (q^{\frac{1}{2}} - q^{\frac{-1}{2}})x^{123,213}. \end{aligned}$$

Thus, $p_{213,213,123}(q) = 1$ and $p_{213,213,213}(q) = q$.

# A combinatorial interpretation

**Theorem**

*Fix $s_{i_1} \cdots s_{i_\ell}$ a reduced expression for $u$, then the coefficient of $q^k$ in $p_{u,v,w}(q)$ is the number of sequences $(\pi^{(0)} = v, \pi^{(1)}, \ldots, \pi^{(\ell)} = w)$ of permutations such that*

1. $\pi^{(j)} \in \{s_{i_j}\pi^{(j-1)}, \pi^{(j-1)}\}$ *for* $j = 1, \ldots, \ell$,
2. $\pi^{(j)} = s_{i_j}\pi^{(j-1)}$ *if* $s_{i_j}\pi^{(j-1)} > \pi^{(j-1)}$,
3. $\pi^{(j)} = \pi^{(j-1)}$ *for exactly* $k$ *values of* $j$.

# Bar Involution

There is a certain map which comes up in the study of quantum groups called the *bar involution*. The bar involution on $\mathcal{A}_{[n],[n]}(n;q)$ is defined by

**Definition**

$$\overline{x_{u_1,v_1}\cdots x_{u_n,v_n}} = x_{u_n,v_n}\cdots x_{u_1,v_1},$$

because $w_0 u = u_n \cdots u_1$ we can say

$$\overline{x^{u,v}} = x^{w_0 u, w_0 v}.$$

# More on Bar Involution

**FACT**

$\{\overline{x^{e,v}} \mid v \in \mathfrak{S}_n\}$ is a basis for the immanant space because it is $\{x^{w_0,w_0 v} \mid v \in \mathfrak{S}_n\}$.

**Example**

$$\overline{x^{123,132}} = x^{321,231}$$

$$= x^{123,132} + (q^{\frac{1}{2}} - q^{\frac{-1}{2}})(x^{123,231} + x^{123,312}) + (q^{\frac{1}{2}} - q^{\frac{-1}{2}})^2 x^{123,321}$$

# Application

**Question**

What is the transition matrix between $\{\overline{x^{e,v}} \mid v \in \mathfrak{S}_n\}$ and $\{x^{e,v} \mid v \in \mathfrak{S}_n\}$?

**Answer**

This is a special case of the previous theorem,

$$\overline{x^{e,v}} = \sum_{w \geq v} p_{w_0, w_0 v, w}(q^{\frac{1}{2}} - q^{\frac{-1}{2}}) x^{e,w}.$$

The bar involution is related to *(inverse) Kazhdan-Lusztig polynomials*.