Inequalities for Symmetric Polynomials

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October 24, 2009

This talk is based on

- "Inequalities for Symmetric Means", with Allison Cuttler, Mark Skandera (to appear in European Jour. Combinatorics).
- "Inequalities for Symmetric Functions of Degree 3", with Jeffrey Kroll, Jonathan Lima, Mark Skandera, and Rengyi Xu (to appear).
- Other work in progress.

Available on request, or at `www.haverford.edu/math/cgreene`*.*

Classical examples (e.g., Hardy-Littlewood-Polya)

THE AGM INEQUALITY:

$$\frac{x_1 + x_2 + \cdots + x_n}{n} \geq (x_1 x_2 \cdots x_n)^{1/n} \quad \forall \mathbf{x} \geq 0.$$

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NEWTON'S INEQUALITIES:

$$\frac{e_k(\mathbf{x})}{e_k(\mathbf{1})} \frac{e_k(\mathbf{x})}{e_k(\mathbf{1})} \geq \frac{e_{k-1}(\mathbf{x})}{e_{k-1}(\mathbf{1})} \frac{e_{k+1}(\mathbf{x})}{e_{k+1}(\mathbf{1})} \quad \forall \mathbf{x} \geq 0$$

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MUIRHEAD'S INEQUALITIES: If $|\lambda| = |\mu|$, then

$$\frac{m_\lambda(\mathbf{x})}{m_\lambda(\mathbf{1})} \geq \frac{m_\mu(\mathbf{x})}{m_\mu(\mathbf{1})} \quad \forall \mathbf{x} \geq 0 \quad \textit{iff}\ \lambda \succeq \mu \ \ (\textit{majorization}).$$

Other examples: different degrees

MACLAURIN'S INEQUALITIES:

$$\left(\frac{e_j(\mathbf{x})}{e_j(\mathbf{1})}\right)^{1/j} \geq \left(\frac{e_k(\mathbf{x})}{e_k(\mathbf{1})}\right)^{1/k} \quad \textit{if } j \leq k,\ \mathbf{x} \geq 0$$

SCHLÖMILCH'S (POWER SUM) INEQUALITIES:

$$\left(\frac{p_j(\mathbf{x})}{n}\right)^{1/j} \leq \left(\frac{p_k(\mathbf{x})}{n}\right)^{1/k} \quad \textit{if } j \leq k,\ \mathbf{x} \geq 0$$

Some results

- Muirhead-like theorems (and conjectures) for all of the classical families.
- A single "master theorem" that includes many of these.
- Proofs based on a new (and potentially interesting) kind of "positivity".

Definitions

We consider two kinds of "averages":

- *Term averages*:

$$F(\mathbf{x}) = \frac{1}{f(\mathbf{1})} f(\mathbf{x}),$$

assuming f has nonnegative integer coefficients. And also

- *Means*:

$$\mathfrak{F}(\mathbf{x}) = \left(\frac{1}{f(\mathbf{1})} f(\mathbf{x})\right)^{1/d}$$

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where f is homogeneous of degree d.

Example:

$$E_k(\mathbf{x}) = \frac{1}{\binom{n}{k}} e_k(\mathbf{x}) \qquad \mathfrak{E}_k(\mathbf{x}) = (E_k(\mathbf{x}))^{1/k}$$

Muirhead-like Inequalities:

ELEMENTARY: $E_\lambda(\mathbf{x}) \geq E_\mu(\mathbf{x}), \ \ \mathbf{x} \geq 0 \iff \lambda \preceq \mu.$

POWER SUM: $P_\lambda(\mathbf{x}) \leq P_\mu(\mathbf{x}), \ \ \mathbf{x} \geq 0 \iff \lambda \preceq \mu.$

HOMOGENEOUS: $H_\lambda(\mathbf{x}) \leq H_\mu(\mathbf{x}), \ \ \mathbf{x} \geq 0 \impliedby \lambda \preceq \mu.$

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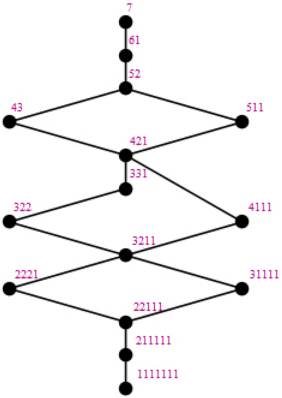
CONJECTURE: the last two implications are $\iff$.

Reference: Cuttler,Greene, Skandera

The Majorization Poset $\mathcal{P}_7$



(Governs term-average inequalities for $E_\lambda, P_\lambda, H_\lambda, S_\lambda$ and M_λ.)

Majorization vs. Normalized Majorization

MAJORIZATION: $\lambda \preceq \mu$ iff $\lambda_1 + \cdots \lambda_i \leq \mu_1 + \cdots \mu_i \; \forall i$

MAJORIZATION POSET: $(\mathcal{P}_n, \preceq)$ on partitions $\lambda \vdash n$.

NORMALIZED MAJORIZATION: $\lambda \sqsubseteq \mu$ iff $\frac{\lambda}{|\lambda|} \preceq \frac{\mu}{|\mu|}$.

NORMALIZED MAJORIZATION POSET: Define $\mathcal{P}_* = \bigcup_n \mathcal{P}_n$. Then $(\overline{\mathcal{P}}_*, \sqsubseteq) =$ quotient of $(\mathcal{P}_*, \sqsubseteq)$ (a preorder) under the relation $\alpha \sim \beta$ if $\alpha \sqsubseteq \beta$ and $\beta \sqsubseteq \alpha$.

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NOTES:

- $(\overline{\mathcal{P}}_*, \sqsubseteq)$ is a lattice, but is not locally finite. $(\overline{\mathcal{P}}_{\leq n}, \sqsubseteq)$ is not a lattice.
- $(\mathcal{P}_n, \preceq)$ embeds in $(\overline{\mathcal{P}}_*, \sqsubseteq)$ as a sublattice and in $(\overline{\mathcal{P}}_{\leq n}, \sqsubseteq)$ as a subposet.

$\overline{\mathcal{P}}_{\leq n} \longleftrightarrow$ *partitions* λ *with* $|\lambda| \leq n$ *whose parts are relatively prime.*

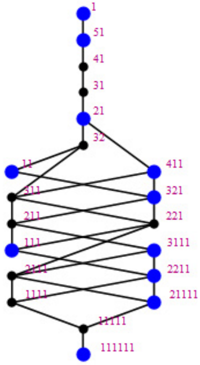

Figure: $(\overline{\mathcal{P}}_{\leq 6}, \sqsubseteq)$ with an embedding of $(\mathcal{P}_6, \preceq)$ shown in blue.

Muirhead-like Inequalities for Means

ELEMENTARY: $\mathfrak{E}_\lambda(\mathbf{x}) \geq \mathfrak{E}_\mu(\mathbf{x}), \quad \mathbf{x} \geq 0 \quad \Longleftrightarrow \quad \lambda \sqsubseteq \mu.$

POWER SUM: $\mathfrak{P}_\lambda(\mathbf{x}) \leq \mathfrak{P}_\mu(\mathbf{x}), \quad \mathbf{x} \geq 0 \quad \Longleftrightarrow \quad \lambda \sqsubseteq \mu.$

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CONJECTURE: the last implication is $\Longleftrightarrow$.

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What about inequalities for Schur means $\mathfrak{S}_\lambda$? We have no idea.

What about inequalities for monomial means $\mathfrak{M}_\lambda$? We know a lot.

A "Master Theorem" for Monomial Means

THEOREM/CONJECTURE: $\mathfrak{M}_\lambda(\mathbf{x}) \leq \mathfrak{M}_\mu(\mathbf{x})$ *iff* $\lambda \trianglelefteq \mu$.
where $\lambda \trianglelefteq \mu$ is the *double majorization order* (to be defined shortly).

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where $\lambda \trianglelefteq \mu$ is the *double majorization order* (to be defined shortly).

Generalizes Muirhead's inequality; allows comparison of symmetric polynomials of different degrees.

The double (normalized) majorization order

DEFINITION: $\lambda \trianglelefteq \mu$ iff $\lambda \sqsubseteq \mu$ and $\lambda^\top \sqsupseteq \mu^\top$,

EQUIVALENTLY: $\lambda \trianglelefteq \mu$ iff $\frac{\lambda}{|\lambda|} \preceq \frac{\mu}{|\mu|}$ and $\frac{\lambda^\top}{|\lambda|} \succeq \frac{\mu^\top}{|\mu|}$.

DEFINITION: $\mathcal{DP}_* = (\mathcal{P}_*, \ \trianglelefteq\ )$

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DEFINITION: $\mathcal{DP}_* = (\mathcal{P}_*, \ \trianglelefteq \ )$

NOTES:

- *The conditions $\lambda \sqsubseteq \mu$ and $\lambda^\top \sqsupseteq \mu^\top$ are not equivalent.* *Example:* $\lambda = \{2, 2\}, \mu = \{2, 1\}$.
- *If $\lambda \trianglelefteq \mu$ and $\mu \trianglelefteq \lambda$, then $\lambda = \mu$; hence $\mathcal{DP}_*$ is a partial order.*
- *$\mathcal{DP}_*$ is self-dual and locally finite, but is not locally ranked, and is not a lattice.*
- *For all n, $(\mathcal{P}_n, \preceq)$ embeds isomorphically in $\mathcal{DP}_*$ as a subposet.*

$\mathcal{DP}_{\leq 5}$

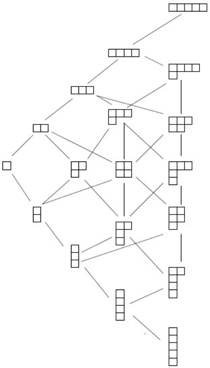

Figure: Double majorization poset $\mathcal{DP}_{\leq 5}$ with vertical embeddings of $\mathcal{P}_n$, $n = 1, 2, \ldots, 5$. (Governs inequalities for $\mathfrak{M}_\lambda$.)

$\mathcal{DP}_{\leq 6}$

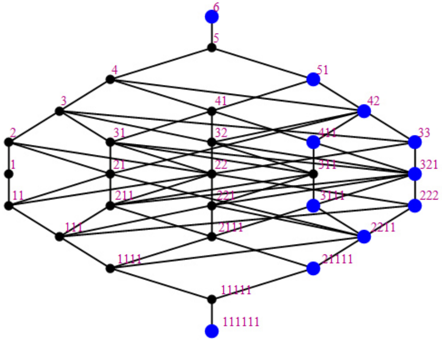

Figure: Double majorization poset $\mathcal{DP}_{\leq 6}$ with an embedding of $\mathcal{P}_6$ shown in blue.

Much of the conjecture has been proved:

"MASTER THEOREM": λ, μ any partitions

$\mathfrak{M}_\lambda \leq \mathfrak{M}_\mu$ if and only if $\lambda \trianglelefteq \mu$, i.e., $\frac{\lambda}{|\lambda|} \preceq \frac{\mu}{|\mu|}$ and $\frac{\lambda^\top}{|\lambda|} \succeq \frac{\mu^\top}{|\mu|}$.

PROVED:

- The "only if" part.
- For all λ, μ with $|\lambda| \leq |\mu|$.
- For λ, μ with $|\lambda|, |\mu| \leq 6$ $(\mathcal{DP}_{\leq 6})$.
- For many other special cases.

Interesting question

The Master Theorem/Conjecture combined with our other results about $\mathfrak{P}_\lambda$ and $\mathfrak{E}_\lambda$ imply the following statement:

$$\mathfrak{M}_\lambda(\mathbf{x}) \leq \mathfrak{M}_\mu(\mathbf{x}) \Leftrightarrow \mathfrak{E}_{\lambda^\top(\mathbf{x})} \leq \mathfrak{E}_{\mu^\top(\mathbf{x})} \ \textit{and} \ \mathfrak{P}_\lambda(\mathbf{x}) \leq \mathfrak{P}_\mu(\mathbf{x}).$$

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Is there a non-combinatorial (e.g., algebraic) proof of this?

Why are these results true? Y-Positivity

ALL of the inequalities in this talk can be established by an argument of the following type:

Assuming that $F(\mathbf{x})$ and $G(\mathbf{x})$ are symmetric polynomials, let $F(\mathbf{y})$ and $G(\mathbf{y})$ be obtained from $F(\mathbf{x})$ and $G(\mathbf{x})$ by making the substitution

$$x_i = y_i + y_{i+1} + \cdots y_n, \quad i = 1, \ldots, n.$$

Then $F(\mathbf{y}) - G(\mathbf{y})$ is a polynomial in $\mathbf{y}$ with nonnegative coefficients. Hence $F(\mathbf{x}) \geq G(\mathbf{x})$ for all $\mathbf{x} \geq 0$.

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We call this phenomenon y-positivity – or maybe it should be "why-positivity" . . . .

Example: AGM Inequality

```
In[1]:= n = 4;
        LHS = (Sum[x[i], {i, n}] / n)^n
        RHS = Product[x[i], {i, n}]
```

Out[2]= $\frac{1}{256}(x[1]+x[2]+x[3]+x[4])^4$

Out[3]= $x[1]\,x[2]\,x[3]\,x[4]$

```
In[4]:= LHS - RHS /. Table[x[i] → Sum[y[j], {j, i, n}], {i, n}]
```

Out[4]= $-y[4]\,(y[3]+y[4])\,(y[2]+y[3]+y[4])\,(y[1]+y[2]+y[3]+y[4]) + \frac{1}{256}(y[1]+2\,y[2]+3\,y[3]+4\,y[4])^4$

```
In[5]:= % // Expand
```

Out[5]= $\frac{y[1]^4}{256} + \frac{1}{32}y[1]^3y[2] + \frac{3}{32}y[1]^2y[2]^2 + \frac{1}{8}y[1]\,y[2]^3 + \frac{y[2]^4}{16} + \frac{3}{64}y[1]^3y[3] + \frac{9}{32}y[1]^2y[2]\,y[3] + \frac{9}{16}y[1]\,y[2]^2y[3] + \frac{3}{8}y[2]^3y[3] + \frac{27}{128}y[1]^2y[3]^2 + \frac{27}{32}y[1]\,y[2]\,y[3]^2 + \frac{27}{32}y[2]^2y[3]^2 + \frac{27}{64}y[1]\,y[3]^3 + \frac{27}{32}y[2]\,y[3]^3 + \frac{81\,y[3]^4}{256} + \frac{1}{16}y[1]^3y[4] + \frac{3}{8}y[1]^2y[2]\,y[4] + \frac{3}{4}y[1]\,y[2]^2y[4] + \frac{1}{2}y[2]^3y[4] + \frac{9}{16}y[1]^2y[3]\,y[4] + \frac{5}{4}y[1]\,y[2]\,y[3]\,y[4] + \frac{5}{4}y[2]^2y[3]\,y[4] + \frac{11}{16}y[1]\,y[3]^2y[4] + \frac{11}{8}y[2]\,y[3]^2y[4] + \frac{11}{16}y[3]^3y[4] + \frac{3}{8}y[1]^2y[4]^2 + \frac{1}{2}y[1]\,y[2]\,y[4]^2 + \frac{1}{2}y[2]^2y[4]^2 + \frac{1}{4}y[1]\,y[3]\,y[4]^2 + \frac{1}{2}y[2]\,y[3]\,y[4]^2 + \frac{3}{8}y[3]^2y[4]^2$

Y-Positivity Conjecture for Schur Functions

If $|\lambda| = |\mu|$ and $\lambda \succeq \mu$, then

$$\frac{s_\lambda(\mathbf{x})}{s_\lambda(\mathbf{1})} - \frac{s_\mu(\mathbf{x})}{s_\mu(\mathbf{1})} \;\Bigg|\; x_i \to y_i + \cdots y_n$$

is a polynomial in $\mathbf{y}$ with nonnegative coefficients.

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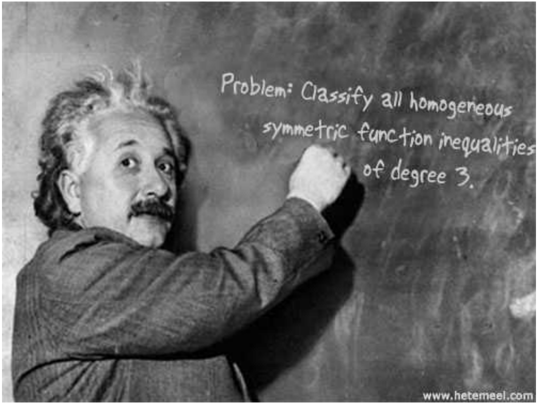
is a polynomial in $\mathbf{y}$ with nonnegative coefficients.

Proved for $|\lambda| \leq 9$ and all n. (CG + Renggyi (Emily) Xu)

"Ultimate" Problem: Classify *all* homogeneous symmetric function inequalities.

More Modest Problem: Classify all homogeneous symmetric function inequalities of degree 3.

This has long been recognized as an important question.



Classifying *all* symmetric function inequalities of degree 3

We seek to characterize symmetric $f(\mathbf{x})$ such that $f(\mathbf{x}) \geq 0$ for all $\mathbf{x} \geq 0$. Such f's will be called *nonnegative*.

- If f is homogeneous of degree 3 then $f(\mathbf{x}) = \alpha m_3(\mathbf{x}) + \beta m_{21}(\mathbf{x}) + \gamma m_{111}(\mathbf{x})$, where the m's are monomial symmetric functions.
- Suppose that $f(\mathbf{x})$ has n variables. Then the correspondence $f \longleftrightarrow (\alpha, \beta, \gamma)$ parameterizes the set of nonnegative f's by a cone in $\mathbb{R}^3$ with n extreme rays.

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- Suppose that $f(\mathbf{x})$ has n variables. Then the correspondence $f \longleftrightarrow (\alpha, \beta, \gamma)$ parameterizes the set of nonnegative f's by a cone in $\mathbb{R}^3$ with n extreme rays. This is not obvious.
- We call it the *positivity cone* $P_{n,3}$. (Structure depends on n.)

Example:

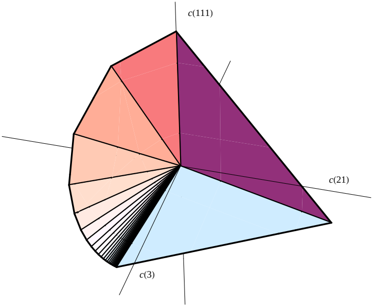
For example, if $n = 3$, there are three extreme rays, spanned by

$$\begin{aligned} f_1(\mathbf{x}) &= m_{21}(\mathbf{x}) - 6m_{111}(\mathbf{x}) \\ f_2(\mathbf{x}) &= m_{111}(\mathbf{x}) \\ f_3(\mathbf{x}) &= m_3(\mathbf{x}) - m_{21}(\mathbf{x}) + 3m_{111}(\mathbf{x}). \end{aligned}$$

If f is cubic, nonnegative, and symmetric in variables $\mathbf{x} = (x_1, x_2, x_3)$ then f may be expressed as a nonnegative linear combination of these three functions.

Example:

If $n = 25$, the cone looks like this:



Main Result:

Theorem: If $\mathbf{x} = (x_1, x_2, \ldots, x_n)$, and $f(\mathbf{x})$ is a symmetric function of degree 3, then $f(\mathbf{x})$ is nonnegative if and only if $f(\mathbf{1}_k^n) \geq 0$ for $k = 1, \ldots, n$, where $\mathbf{1}_k^n = (1, \ldots, 1, 0, \ldots, 0)$, with k ones and $(n-k)$ zeros.

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Example: If $\mathbf{x} = (x_1, x_2, x_3)$ and $f(\mathbf{x}) = m_3(\mathbf{x}) - m_{21}(\mathbf{x}) + 3m_{111}(\mathbf{x})$, then

$$\begin{aligned} f(1,0,0) &= 1 \\ f(1,1,0) &= 2-2=0 \\ f(1,1,1) &= 3-6+3=0 \end{aligned}$$

Main Result:

Theorem: If $\mathbf{x} = (x_1, x_2, \ldots, x_n)$, and $f(\mathbf{x})$ is a symmetric function of degree 3, then $f(\mathbf{x})$ is nonnegative if and only if $f(\mathbf{1}_k^n) \geq 0$ for $k = 1, \ldots, n$, where $\mathbf{1}_k^n = (1, \ldots, 1, 0, \ldots, 0)$, with k ones and $(n-k)$ zeros.

Example: If $\mathbf{x} = (x_1, x_2, x_3)$ and $f(\mathbf{x}) = m_3(\mathbf{x}) - m_{21}(\mathbf{x}) + 3m_{111}(\mathbf{x})$, then

$$\begin{aligned} f(1,0,0) &= 1 \\ f(1,1,0) &= 2-2=0 \\ f(1,1,1) &= 3-6+3=0 \end{aligned}$$

NOTES:

- The inequality $f(\mathbf{x}) \geq 0$ is known as *Schur's Inequality* (HLP).
- The statement analogous to the above theorem for degree $d > 3$ is false.

Application: A positive function that is not y-positive

Again take $f(\mathbf{x}) = m_3(\mathbf{x}) - m_{21}(\mathbf{x}) + 3m_{111}(\mathbf{x})$, but with $n = 5$ variables.

Then $f(1,0,0,0,0) = 1$, $f(1,1,0,0,0) = 0$, $f(1,1,1,0,0) = 0$, $f(1,1,1,1,0) = 4$, $f(1,1,1,1,1) = 15$. Hence, by the Theorem, $f(\mathbf{x}) \geq 0$ for all $\mathbf{x} \geq 0$.

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However, y-substitution give $f(\mathbf{y}) =$

Out[12]= $y[1]^3 + 2 y[1]^2 y[2] + y[1]^2 y[3] + y[1] y[2] y[3] + y[2]^2 y[3] + 2 y[1] y[2] y[4] + 2 y[2]^2 y[4] + 4 y[1] y[3] y[4] + 8 y[2] y[3] y[4] + 6 y[3]^2 y[4] + 3 y[1] y[4]^2 + 6 y[2] y[4]^2 + 9 y[3] y[4]^2 + 4 y[4]^3 - y[1]^2 y[5] + 3 y[1] y[2] y[5] + 3 y[2]^2 y[5] + 8 y[1] y[3] y[5] + 16 y[2] y[3] y[5] + 12 y[3]^2 y[5] + 13 y[1] y[4] y[5] + 26 y[2] y[4] y[5] + 39 y[3] y[4] y[5] + 26 y[4]^2 y[5] + 9 y[1] y[5]^2 + 18 y[2] y[5]^2 + 27 y[3] y[5]^2 + 36 y[4] y[5]^2 + 15 y[5]^3$

which has exactly one negative coefficient, $-y[1]^2 y[5]$.

Reference:

- "Inequalities for Symmetric Functions of Degree 3", with Jeffrey Kroll, Jonathan Lima, Mark Skandera, and Rengyi Xu (to appear).

Available on request, or at `www.haverford.edu/math/cgreene`*.*

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