Derangements and Cubes

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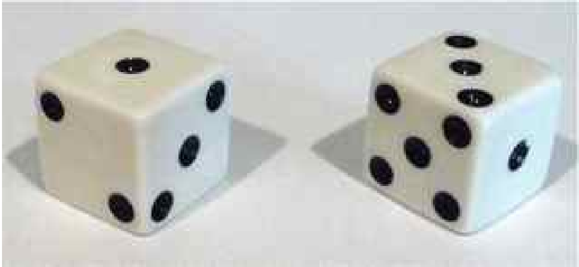
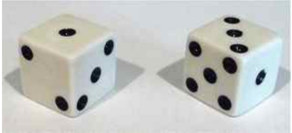
Joint work with Liz McMahon

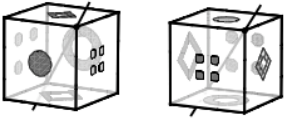
Problem How many ways can you roll a die so that *none* of its faces are in the same position?

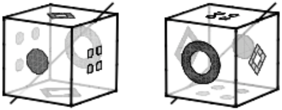


Before After

Problem How many ways can you roll a die so that *none* of its faces are in the same position?







8 vertex rotations

6 edge rotations

Direct Isometries corresponding to face derangements

Answer: 14

Derangements

Hatcheck Problem How many ways can we return n hats to n people so that no one receives her own hat?

A *derangement* of a set S is a permutation with no fixed points.

Theorem

The number of derangements $d_n = n! \sum_{k=0}^{n} \frac{(-1)^k}{k!}$. *Thus,* $d_n/n! \to e^{-1} \approx 0.367879\ldots$

Theorem

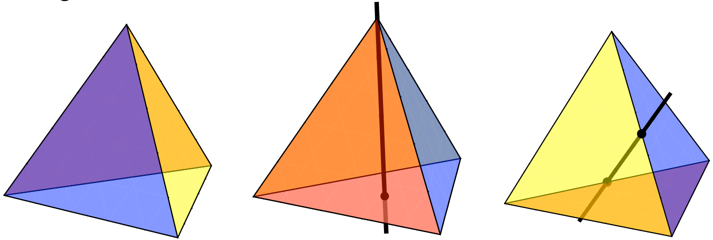
Recursion: $d_n = (n-1)(d_{n-1} + d_{n-2})$

Geometry of derangements

Geometric Fact

Derangements of $[n]$ $\leftrightarrow$ isometries of the regular $(n-1)$-simplex in which every one of the n facets is moved.

In $\mathbb{R}^3$, regular tetrahedron has 4! isometries – Rotations



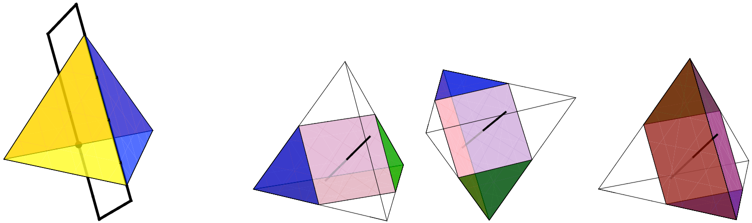
Identity   Face rotations(8)   Edge rotations (3)

Geometry of derangements

Geometric Fact

Derangements of $[n] \leftrightarrow$ isometries of the regular $(n-1)$-simplex in which every one of the n facets is moved.

Reflections and rotary reflections



Reflections (6) Rotary reflections (6)

Derangements 3 edge rotations and 6 rotary reflections: $d_4 = 9$

Cubes and coats

Couples Coatcheck Problem *n couples* each check their two coats at the beginning of a party; the attendant puts a couple's 2 coats on a single hanger.

- Attendant randomly selects a hanger;
- Attendant randomly hands a coat from that hanger to each person in the couple.

How many ways can the coats be returned so that no one gets their own coat back?

Cubes and coats

Definition

c-derangements: Let $\hat{d}_n$ be the number of ways to return the coats so that no one receives their own coat.

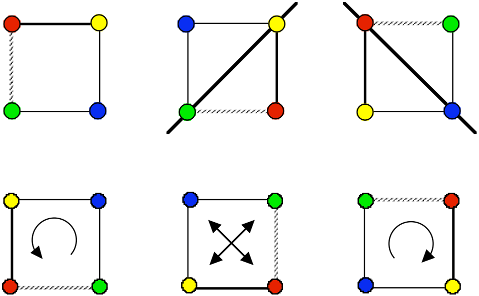
Facts:

- There are $2^n n!$ ways to return the $2n$ coats.
- There are $2^n n!$ isometries of an n-cube.
- The number of coat derangements $\hat{d}_n$ is the same as the number of facet derangements of the n-cube.

Squares

Deranging the edges of a square.

$\hat{d}_2 = 5$

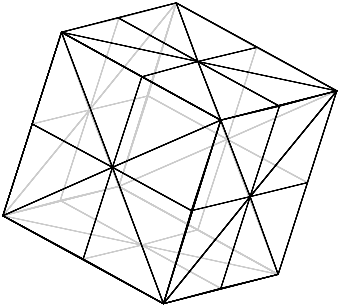


The 5 edge derangements of a square.

Isometries of the cube

Fact: There are $2^3 3! = 48$ isometries of a cube.

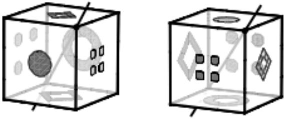
- Direct
  - The identity;
  - 8 vertex rotations of 120° and 240°;
  - 6 180° edge rotations;
  - 9 rotations through the centers of opposite faces.
- Indirect
  - 9 reflections
  - 15 rotary reflections



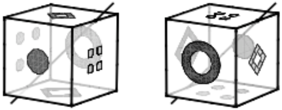
Direct face derangements

Direct isometries

- The identity;
- 8 vertex rotations of 120° and 240°;
- 6 180° edge rotations;
- 9 rotations through the centers of opposite faces.



8 vertex rotations

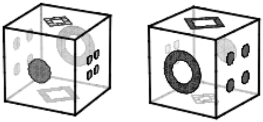


6 edge rotations

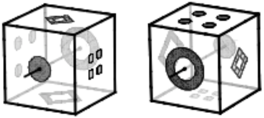
Direct Isometries corresponding to face derangements

Indirect face derangements

Central inversion ($z \leftrightarrow -z$)

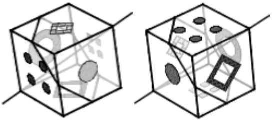


Reducible rotary reflection (6)



Irreducible rotary reflection (8)

$\hat{d}_3 = 14 + 15 = 29$



Formulas

Theorem

Let $\hat{d}_n$ be the number of facet derangements of the n-cube.

- $\hat{d}_n = 2^n n! \sum_{k=0}^{n} \frac{(-1)^k}{2^k k!}$ *Compare:* $d_n = n! \sum_{k=0}^{n} \frac{(-1)^k}{k!}$
- $\hat{d}_n = \sum_{k=0}^{n} \binom{n}{k} 2^k d_k$, *where* $d_n =$ *(ordinary) derangements.*
- *Recursion:* $\hat{d}_n = (2n-1)\hat{d}_{n-1} + (2n-2)\hat{d}_{n-2}$

  Compare: $d_n = (n-1)(d_{n-1} + d_{n-2})$

Data

Probabilistic interpretation

In the coatcheck problem, the probability that no one receives their own coat approaches $e^{-1/2} \approx 0.6065\ldots$ *as* $n \to \infty$.
[Compare: $d_n \to e^{-1} \approx 0.3679\ldots$*]*

Derangement numbers

n	0	1	2	3	4	5	6
d_n	1	0	1	2	9	44	265
$\hat{d}_n$	1	1	5	29	233	2329	27,949

Rates of convergence

$$\frac{d_6}{6!} - \frac{1}{e} = 1.76 \times 10^{-4} \qquad \frac{\hat{d}_6}{2^6 6!} - \frac{1}{\sqrt{e}} = 1.46 \times 10^{-6}$$

More data

Ordinary derangements

Direct isometries $\leftrightarrow$ even permutations
Indirect isometries $\leftrightarrow$ odd permutations

Number of even and odd derangements for $n \leq 7$.

n	1	2	3	4	5	6	7
d_n	0	1	2	9	44	265	1854
e_n	0	0	2	3	24	130	930
o_n	0	1	0	6	20	135	924
$e_n - o_n$	0	-1	2	-3	4	-5	6

More more data

Hypercube facet derangements

Direct isometries $\leftrightarrow$ 'even' permutations
Indirect isometries $\leftrightarrow$ 'odd' permutations

Number of even and odd hypercube derangements for $n \leq 7$.

n	1	2	3	4	5	6	7
$\hat{d}_n$	1	5	29	233	2329	27,949	391,285
$\hat{e}_n$	0	3	14	117	1164	13,975	195,642
$\hat{o}_n$	1	2	15	116	1165	13,974	195,643
$\hat{e}_n - \hat{o}_n$	-1	1	-1	1	-1	1	-1

Direct and indirect facet derangements

Theorem

Let $\hat{e}_n$ and $\hat{o}_n$ be the number of direct and indirect facet derangements of a cube, resp. Then

$$\hat{e}_n - \hat{o}_n = (-1)^n.$$

Proof idea

- Each facet derangement $\leftrightarrow$ signed permutation matrix.

$$A = \begin{pmatrix} -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 & 0 \end{pmatrix} \leftrightarrow (11^*)(22^*)(345^*)(3^*4^*5)$$

$\hat{e}_n - \hat{o}_n = (-1)^n$.

- Easy fact: $det(A) = \pm 1$.
- An isometry is direct iff $det(A) = 1$.
- Find the first row k with $a_{k,k} = 0$.

$$A = \begin{pmatrix} -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 & 0 \end{pmatrix}$$

$\hat{e}_n - \hat{o}_n = (-1)^n$.

- Change the sign of the only non-zero entry in row k to produce a new matrix A':

$$A \qquad\qquad A'$$

$$\begin{pmatrix} -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 & 0 \end{pmatrix} \begin{pmatrix} -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 & 0 \end{pmatrix}$$

$$A \leftrightarrow (11^*)(22^*)(345^*)(3^*4^*5)$$

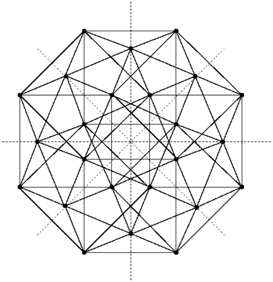
$$A' \leftrightarrow (11^*)(22^*)(34^*53^*45^*)$$

$\hat{e}_n - \hat{o}_n = (-1)^n$.

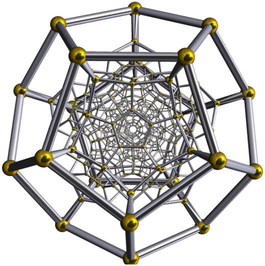
In this example, A is direct and A' is indirect. In general, this involution (almost) gives a 1-1 correspondence between direct and indirect facet-derangements.

- Central inversion $\leftrightarrow$ the matrix $-I$.
- n even $\leftrightarrow$ central inversion is direct.
- n odd $\leftrightarrow$ central inversion is indirect.

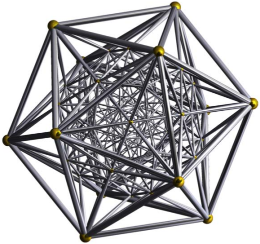
Future projects - 4 dimensions



24-cell



120-cell



600-cell

- Find the number of vertex, edge, 2-dimensional and 3-dimensional face derangement numbers for the 24-cell and the 120-cell.
- For each class of derangements, count the direct and indirect isometries.