Descent sets of cyclic permutations

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Permutations

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The *descent set* of $\pi \in \mathcal{S}_n$ is

$$D(\pi) = \{i \,:\, 1 \leq i \leq n-1,\ \pi(i) > \pi(i+1)\}.$$

$$D(25{\cdot}17{\cdot}36{\cdot}4) = \{2, 4, 6\}$$

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Descents of cyclic permutations come up when determining the smallest number of symbols needed to realize a permutation by shifts.

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E.g., the permutation 4217536 can be realized using three symbols:

$$\left.\begin{array}{rc} 2102212210\ldots & 4 \\ 102212210\ldots & 2 \\ 02212210\ldots & 1 \\ 2212210\ldots & 7 \\ 212210\ldots & 5 \\ 12210\ldots & 3 \\ 2210\ldots & 6 \end{array}\right\} \text{lexicographic order of the shifted sequences}$$

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The number of symbols needed is related to the descents of the cycle $(4, 2, 1, 7, 5, 3, 6)$.

Descent sets of 5-cycles

$\mathcal{C}_5$	
$(1,2,3,4,5) = 2345{\cdot}1$	
$(2,1,3,4,5) = 3{\cdot}145{\cdot}2$ $(3,2,1,4,5) = 4{\cdot}125{\cdot}3$ $(4,3,2,1,5) = 5{\cdot}1234$	
$(1,3,2,4,5) = 34{\cdot}25{\cdot}1$ $(1,4,3,2,5) = 45{\cdot}23{\cdot}1$ $(3,1,2,4,5) = 24{\cdot}15{\cdot}3$ $(3,1,4,2,5) = 45{\cdot}123$ $(4,3,1,2,5) = 25{\cdot}134$	
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Main theorem

Theorem

For every n there is a bijection $\varphi : \mathcal{C}_{n+1} \to \mathcal{S}_n$ *such that if* $\pi \in \mathcal{C}_{n+1}$ *and* $\sigma = \varphi(\pi)$*, then*

$$D(\pi) \cap [n-1] = D(\sigma).$$

The bijection $\varphi : \mathcal{C}_{n+1} \longrightarrow \mathcal{S}_n$. First step

Given $\pi \in \mathcal{C}_{n+1}$, write it in cycle form with $n+1$ at the end:

$\pi = (11, 4, 10, 1, 7, 16, 9, 3, 5, 12, 20, 2, 6, 14, 18, 8, 13, 19, 15, 17, 21) \in \mathcal{C}_{21}$

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Delete $n+1$ and split at the "left-to-right maxima":

$$\sigma = (\underline{11}, 4, 10, 1, 7)(\underline{16}, 9, 3, 5, 12)(\underline{20}, 2, 6, 14, 18, 8, 13, 19, 15, 17) \in \mathcal{S}_{20}.$$

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This map $\pi \mapsto \sigma$ is a bijection, but unfortunately it does not always preserve the descent set:

$$\pi(7) = 16 > \pi(8) = 13 \quad \text{but} \quad \sigma(7) = 11 < \sigma(8) = 13.$$

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We say that the pair $\{7, 8\}$ is *bad*. We will fix the bad pairs.

The bijection $\varphi : \mathcal{C}_{n+1} \to \mathcal{S}_n$. Fixing bad pairs

$$\pi = (11, 4, 10, 1, 7, 16, 9, 3, 5, 12, 20, 2, 6, 14, 18, 8, 13, 19, 15, 17, 21)$$
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For each but the last cycle of σ, from left to right:

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$\{7, 6\}$ and $\{7, 8\}$ are bad; and $\sigma(6) = 14 > 13 = \sigma(8) \Rightarrow \ \varepsilon := -1$.

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Switch 7 and 6.

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For each but the last cycle of σ, from left to right:

- $z :=$ rightmost entry of the cycle.
  If $\{z, z-1\}$ or $\{z, z+1\}$ are bad, let $\varepsilon = \pm 1$ be such that $\{z, z+\varepsilon\}$ is bad and $\sigma(z+\varepsilon)$ is largest.
- Repeat for as long as $\{z, z+\varepsilon\}$ is bad:
  1. Switch z and $z+\varepsilon$ (in the cycle form of σ).
  2. If the elements preceding the last switched entries have consecutive values, switch them. Repeat 2.
  3. $z :=$ new rightmost entry of the cycle.

$$\pi = (11, 4, 10, 1, 7, 16, 9, 3, 5, 12, 20, 2, 6, 14, 18, 8, 13, 19, 15, 17, 21)$$
$$\sigma = (11, 4, 9, 3, 5)(16, 10, 2, 6, 12)(20, 1, 7, 14, 18, 8, 13, 19, 15, 17)$$

$z := 5$. $\varepsilon := -1$.

The bijection $\varphi : \mathcal{C}_{n+1} \to \mathcal{S}_n$. Fixing bad pairs

For each but the last cycle of σ, from left to right:

- $z :=$ rightmost entry of the cycle.
  If $\{z, z-1\}$ or $\{z, z+1\}$ are bad, let $\varepsilon = \pm 1$ be such that $\{z, z+\varepsilon\}$ is bad and $\sigma(z+\varepsilon)$ is largest.
- Repeat for as long as $\{z, z+\varepsilon\}$ is bad:
  1. Switch z and $z+\varepsilon$ (in the cycle form of σ).
  2. If the elements preceding the last switched entries have consecutive values, switch them. Repeat 2.
  3. $z :=$ new rightmost entry of the cycle.

$$\pi = (11, 4, 10, 1, 7, 16, 9, 3, 5, 12, 20, 2, 6, 14, 18, 8, 13, 19, 15, 17, 21)$$
$$\sigma = (11, 4, 9, 3, 5)(16, 10, 2, 6, 12)(20, 1, 7, 14, 18, 8, 13, 19, 15, 17)$$

$z := 5$. $\{5, 4\}$ is OK, so we move on to the second cycle.

The bijection $\varphi : \mathcal{C}_{n+1} \to \mathcal{S}_n$. Fixing bad pairs

For each but the last cycle of σ, from left to right:

- $z :=$ rightmost entry of the cycle.
  If $\{z, z-1\}$ or $\{z, z+1\}$ are bad, let $\varepsilon = \pm 1$ be such that $\{z, z+\varepsilon\}$ is bad and $\sigma(z+\varepsilon)$ is largest.
- Repeat for as long as $\{z, z+\varepsilon\}$ is bad:
  1. Switch z and $z+\varepsilon$ (in the cycle form of σ).
  2. If the elements preceding the last switched entries have consecutive values, switch them. Repeat 2.
  3. $z :=$ new rightmost entry of the cycle.

$$\pi = (11, 4, 10, 1, 7, 16, 9, 3, 5, 12, 20, 2, 6, 14, 18, 8, 13, 19, 15, 17, 21)$$
$$\sigma = (11, 4, 9, 3, 5)(16, 10, 2, 6, 12)(20, 1, 7, 14, 18, 8, 13, 19, 15, 17)$$

$z := 12$.

The bijection $\varphi : \mathcal{C}_{n+1} \to \mathcal{S}_n$. Fixing bad pairs

For each but the last cycle of σ, from left to right:

- $z :=$ rightmost entry of the cycle.
  If $\{z, z-1\}$ or $\{z, z+1\}$ are bad, let $\varepsilon = \pm 1$ be such that $\{z, z+\varepsilon\}$ is bad and $\sigma(z+\varepsilon)$ is largest.
- Repeat for as long as $\{z, z+\varepsilon\}$ is bad:
  1. Switch z and $z+\varepsilon$ (in the cycle form of σ).
  2. If the elements preceding the last switched entries have consecutive values, switch them. Repeat 2.
  3. $z :=$ new rightmost entry of the cycle.

$$\pi = (11, 4, 10, 1, 7, 16, 9, 3, 5, 12, 20, 2, 6, 14, 18, 8, 13, 19, 15, 17, 21)$$
$$\sigma = (11, 4, 9, 3, 5)(16, 10, 2, 6, 12)(20, 1, 7, 14, 18, 8, 13, 19, 15, 17)$$

$z := 12$. $\{12, 11\}$ is OK but $\{12, 13\}$ is bad $\Rightarrow \varepsilon := 1$.

The bijection $\varphi : \mathcal{C}_{n+1} \to \mathcal{S}_n$. Fixing bad pairs

For each but the last cycle of σ, from left to right:

- $z :=$ rightmost entry of the cycle.
  If $\{z, z-1\}$ or $\{z, z+1\}$ are bad, let $\varepsilon = \pm 1$ be such that $\{z, z+\varepsilon\}$ is bad and $\sigma(z+\varepsilon)$ is largest.
- Repeat for as long as $\{z, z+\varepsilon\}$ is bad:
  1. Switch z and $z+\varepsilon$ (in the cycle form of σ).
  2. If the elements preceding the last switched entries have consecutive values, switch them. Repeat 2.
  3. $z :=$ new rightmost entry of the cycle.

$$\pi = (11, 4, 10, 1, 7, 16, 9, 3, 5, 12, 20, 2, 6, 14, 18, 8, 13, 19, 15, 17, 21)$$
$$\sigma = (11, 4, 9, 3, 5)(16, 10, 2, 6, 12)(20, 1, 7, 14, 18, 8, 13, 19, 15, 17)$$

$z := 12$. $\varepsilon := 1$.

Switch 12 and 13.

The bijection $\varphi : \mathcal{C}_{n+1} \to \mathcal{S}_n$. Fixing bad pairs

For each but the last cycle of σ, from left to right:

- $z :=$ rightmost entry of the cycle.
  If $\{z, z-1\}$ or $\{z, z+1\}$ are bad, let $\varepsilon = \pm 1$ be such that $\{z, z+\varepsilon\}$ is bad and $\sigma(z+\varepsilon)$ is largest.
- Repeat for as long as $\{z, z+\varepsilon\}$ is bad:
  1. Switch z and $z+\varepsilon$ (in the cycle form of σ).
  2. If the elements preceding the last switched entries have consecutive values, switch them. Repeat 2.
  3. $z :=$ new rightmost entry of the cycle.

$$\pi = (11, 4, 10, 1, 7, 16, 9, 3, 5, 12, 20, 2, 6, 14, 18, 8, 13, 19, 15, 17, 21)$$
$$\sigma = (11, 4, 9, 3, 5)(16, 10, 2, 6, 13)(20, 1, 7, 14, 18, 8, 12, 19, 15, 17)$$

$z := 12$. $\varepsilon := 1$.

Switch 12 and 13.

The bijection $\varphi : \mathcal{C}_{n+1} \to \mathcal{S}_n$. Fixing bad pairs

For each but the last cycle of σ, from left to right:

- $z :=$ rightmost entry of the cycle.
  If $\{z, z-1\}$ or $\{z, z+1\}$ are bad, let $\varepsilon = \pm 1$ be such that $\{z, z+\varepsilon\}$ is bad and $\sigma(z+\varepsilon)$ is largest.
- Repeat for as long as $\{z, z+\varepsilon\}$ is bad:
  1. Switch z and $z+\varepsilon$ (in the cycle form of σ).
  2. If the elements preceding the last switched entries have consecutive values, switch them. Repeat 2.
  3. $z :=$ new rightmost entry of the cycle.

$$\pi = (11, 4, 10, 1, 7, 16, 9, 3, 5, 12, 20, 2, 6, 14, 18, 8, 13, 19, 15, 17, 21)$$
$$\sigma = (11, 4, 9, 3, 5)(16, 10, 2, \underline{6}, 13)(20, 1, 7, 14, 18, \underline{8}, 12, 19, 15, 17)$$

$z := 12$. $\varepsilon := 1$.

Switch 12 and 13.

The bijection $\varphi : \mathcal{C}_{n+1} \to \mathcal{S}_n$. Fixing bad pairs

For each but the last cycle of σ, from left to right:

- $z :=$ rightmost entry of the cycle.
  If $\{z, z-1\}$ or $\{z, z+1\}$ are bad, let $\varepsilon = \pm 1$ be such that $\{z, z+\varepsilon\}$ is bad and $\sigma(z+\varepsilon)$ is largest.
- Repeat for as long as $\{z, z+\varepsilon\}$ is bad:
  1. Switch z and $z+\varepsilon$ (in the cycle form of σ).
  2. If the elements preceding the last switched entries have consecutive values, switch them. Repeat 2.
  3. $z :=$ new rightmost entry of the cycle.

$$\pi = (11, 4, 10, 1, 7, 16, 9, 3, 5, 12, 20, 2, 6, 14, 18, 8, 13, 19, 15, 17, 21)$$
$$\sigma = (11, 4, 9, 3, 5)(16, 10, 2, 6, 13)(20, 1, 7, 14, 18, 8, 12, 19, 15, 17)$$

$z := 13$. $\varepsilon := 1$.

The bijection $\varphi : \mathcal{C}_{n+1} \to \mathcal{S}_n$. Fixing bad pairs

For each but the last cycle of σ, from left to right:

- $z :=$ rightmost entry of the cycle.
  If $\{z, z-1\}$ or $\{z, z+1\}$ are bad, let $\varepsilon = \pm 1$ be such that $\{z, z+\varepsilon\}$ is bad and $\sigma(z+\varepsilon)$ is largest.
- Repeat for as long as $\{z, z+\varepsilon\}$ is bad:
  1. Switch z and $z+\varepsilon$ (in the cycle form of σ).
  2. If the elements preceding the last switched entries have consecutive values, switch them. Repeat 2.
  3. $z :=$ new rightmost entry of the cycle.

$$\pi = (11, 4, 10, 1, 7, 16, 9, 3, 5, 12, 20, 2, 6, 14, 18, 8, 13, 19, 15, 17, 21)$$
$$\sigma = (11, 4, 9, 3, 5)(16, 10, 2, 6, 13)(20, 1, 7, 14, 18, 8, 12, 19, 15, 17)$$

$z := 13$. $\{13, 14\}$ is bad. $\varepsilon := 1$.

The bijection $\varphi : \mathcal{C}_{n+1} \to \mathcal{S}_n$. Fixing bad pairs

For each but the last cycle of σ, from left to right:

- z := rightmost entry of the cycle.
  If $\{z, z-1\}$ or $\{z, z+1\}$ are bad, let $\varepsilon = \pm 1$ be such that $\{z, z+\varepsilon\}$ is bad and $\sigma(z+\varepsilon)$ is largest.
- Repeat for as long as $\{z, z+\varepsilon\}$ is bad:
  1. Switch z and $z+\varepsilon$ (in the cycle form of σ).
  2. If the elements preceding the last switched entries have consecutive values, switch them. Repeat 2.
  3. z := new rightmost entry of the cycle.

$$\pi = (11, 4, 10, 1, 7, 16, 9, 3, 5, 12, 20, 2, 6, 14, 18, 8, 13, 19, 15, 17, 21)$$
$$\sigma = (11, 4, 9, 3, 5)(16, 10, 2, 6, 13)(20, 1, 7, 14, 18, 8, 12, 19, 15, 17)$$

$z := 13$. $\varepsilon := 1$.

Switch 13 and 14.

The bijection $\varphi : \mathcal{C}_{n+1} \to \mathcal{S}_n$. Fixing bad pairs

For each but the last cycle of σ, from left to right:

- $z :=$ rightmost entry of the cycle. If $\{z, z-1\}$ or $\{z, z+1\}$ are bad, let $\varepsilon = \pm 1$ be such that $\{z, z+\varepsilon\}$ is bad and $\sigma(z+\varepsilon)$ is largest.
- Repeat for as long as $\{z, z+\varepsilon\}$ is bad:
  1. Switch z and $z+\varepsilon$ (in the cycle form of σ).
  2. If the elements preceding the last switched entries have consecutive values, switch them. Repeat 2.
  3. $z :=$ new rightmost entry of the cycle.

$$\pi = (11, 4, 10, 1, 7, 16, 9, 3, 5, 12, 20, 2, 6, 14, 18, 8, 13, 19, 15, 17, 21)$$
$$\sigma = (11, 4, 9, 3, 5)(16, 10, 2, 6, 14)(20, 1, 7, 13, 18, 8, 12, 19, 15, 17)$$

$z := 13$. $\varepsilon := 1$.

Switch 13 and 14.

The bijection $\varphi : \mathcal{C}_{n+1} \to \mathcal{S}_n$. Fixing bad pairs

For each but the last cycle of σ, from left to right:

- $z :=$ rightmost entry of the cycle.
  If $\{z, z-1\}$ or $\{z, z+1\}$ are bad, let $\varepsilon = \pm 1$ be such that $\{z, z+\varepsilon\}$ is bad and $\sigma(z+\varepsilon)$ is largest.
- Repeat for as long as $\{z, z+\varepsilon\}$ is bad:
  1. Switch z and $z+\varepsilon$ (in the cycle form of σ).
  2. If the elements preceding the last switched entries have consecutive values, switch them. Repeat 2.
  3. $z :=$ new rightmost entry of the cycle.

$$\pi = (11, 4, 10, 1, 7, 16, 9, 3, 5, 12, 20, 2, 6, 14, 18, 8, 13, 19, 15, 17, 21)$$
$$\sigma = (11, 4, 9, 3, 5)(16, 10, 2, 6, 14)(20, 1, 7, 13, 18, 8, 12, 19, 15, 17)$$

$z := 13$. $\varepsilon := 1$.

Switch 13 and 14. Switch 6 and 7.

The bijection $\varphi : \mathcal{C}_{n+1} \longrightarrow \mathcal{S}_n$. Fixing bad pairs

For each but the last cycle of σ, from left to right:

- $z :=$ rightmost entry of the cycle.
  If $\{z, z-1\}$ or $\{z, z+1\}$ are bad, let $\varepsilon = \pm 1$ be such that $\{z, z+\varepsilon\}$ is bad and $\sigma(z+\varepsilon)$ is largest.
- Repeat for as long as $\{z, z+\varepsilon\}$ is bad:
  1. Switch z and $z+\varepsilon$ (in the cycle form of σ).
  2. If the elements preceding the last switched entries have consecutive values, switch them. Repeat 2.
  3. $z :=$ new rightmost entry of the cycle.

$$\pi = (11, 4, 10, 1, 7, 16, 9, 3, 5, 12, 20, 2, 6, 14, 18, 8, 13, 19, 15, 17, 21)$$
$$\sigma = (11, 4, 9, 3, 5)(16, 10, 2, 7, 14)(20, 1, 6, 13, 18, 8, 12, 19, 15, 17)$$

$z := 13$. $\varepsilon := 1$.

Switch 13 and 14. Switch 6 and 7.

The bijection $\varphi : \mathcal{C}_{n+1} \to \mathcal{S}_n$. Fixing bad pairs

For each but the last cycle of σ, from left to right:

- $z :=$ rightmost entry of the cycle.
  If $\{z, z-1\}$ or $\{z, z+1\}$ are bad, let $\varepsilon = \pm 1$ be such that $\{z, z+\varepsilon\}$ is bad and $\sigma(z+\varepsilon)$ is largest.
- Repeat for as long as $\{z, z+\varepsilon\}$ is bad:
  1. Switch z and $z+\varepsilon$ (in the cycle form of σ).
  2. If the elements preceding the last switched entries have consecutive values, switch them. Repeat 2.
  3. $z :=$ new rightmost entry of the cycle.

$$\pi = (11, 4, 10, 1, 7, 16, 9, 3, 5, 12, 20, 2, 6, 14, 18, 8, 13, 19, 15, 17, 21)$$

$$\sigma = (11, 4, 9, 3, 5)(16, 10, 2, 7, 14)(20, 1, 6, 13, 18, 8, 12, 19, 15, 17)$$

$z := 13$. $\varepsilon := 1$.

Switch 13 and 14. Switch 6 and 7. Switch 2 and 1.

The bijection $\varphi : \mathcal{C}_{n+1} \longrightarrow \mathcal{S}_n$. Fixing bad pairs

For each but the last cycle of σ, from left to right:

- $z :=$ rightmost entry of the cycle.
  If $\{z, z-1\}$ or $\{z, z+1\}$ are bad, let $\varepsilon = \pm 1$ be such that $\{z, z+\varepsilon\}$ is bad and $\sigma(z+\varepsilon)$ is largest.
- Repeat for as long as $\{z, z+\varepsilon\}$ is bad:
  1. Switch z and $z+\varepsilon$ (in the cycle form of σ).
  2. If the elements preceding the last switched entries have consecutive values, switch them. Repeat 2.
  3. $z :=$ new rightmost entry of the cycle.

$$\pi = (11, 4, 10, 1, 7, 16, 9, 3, 5, 12, 20, 2, 6, 14, 18, 8, 13, 19, 15, 17, 21)$$
$$\sigma = (11, 4, 9, 3, 5)(16, \underline{10}, 1, 7, 14)(\underline{20}, 2, 6, 13, 18, 8, 12, 19, 15, 17)$$

$z := 13$. $\varepsilon := 1$.

Switch 13 and 14. Switch 6 and 7. Switch 2 and 1.

The bijection $\varphi : \mathcal{C}_{n+1} \to \mathcal{S}_n$. Fixing bad pairs

For each but the last cycle of σ, from left to right:

- $z :=$ rightmost entry of the cycle.
  If $\{z, z-1\}$ or $\{z, z+1\}$ are bad, let $\varepsilon = \pm 1$ be such that $\{z, z+\varepsilon\}$ is bad and $\sigma(z+\varepsilon)$ is largest.
- Repeat for as long as $\{z, z+\varepsilon\}$ is bad:
  1. Switch z and $z+\varepsilon$ (in the cycle form of σ).
  2. If the elements preceding the last switched entries have consecutive values, switch them. Repeat 2.
  3. $z :=$ new rightmost entry of the cycle.

$$\pi = (11, 4, 10, 1, 7, 16, 9, 3, 5, 12, 20, 2, 6, 14, 18, 8, 13, 19, 15, 17, 21)$$
$$\sigma = (11, 4, 9, 3, 5)(16, 10, 1, 7, 14)(20, 2, 6, 13, 18, 8, 12, 19, 15, 17)$$

$z := 14$. $\varepsilon := 1$.

The bijection $\varphi : \mathcal{C}_{n+1} \to \mathcal{S}_n$. Fixing bad pairs

For each but the last cycle of σ, from left to right:

- $z :=$ rightmost entry of the cycle.
  If $\{z, z-1\}$ or $\{z, z+1\}$ are bad, let $\varepsilon = \pm 1$ be such that $\{z, z+\varepsilon\}$ is bad and $\sigma(z+\varepsilon)$ is largest.
- Repeat for as long as $\{z, z+\varepsilon\}$ is bad:
  1. Switch z and $z+\varepsilon$ (in the cycle form of σ).
  2. If the elements preceding the last switched entries have consecutive values, switch them. Repeat 2.
  3. $z :=$ new rightmost entry of the cycle.

$$\pi = (11, 4, 10, 1, 7, 16, 9, 3, 5, 12, 20, 2, 6, 14, 18, 8, 13, 19, 15, 17, 21)$$
$$\sigma = (11, 4, 9, 3, 5)(16, 10, 1, 7, 14)(20, 2, 6, 13, 18, 8, 12, 19, 15, 17)$$

$z := 14$. $\{14, 15\}$ is bad. $\varepsilon := 1$.

The bijection $\varphi : \mathcal{C}_{n+1} \to \mathcal{S}_n$. Fixing bad pairs

For each but the last cycle of σ, from left to right:

- $z :=$ rightmost entry of the cycle.
  If $\{z, z-1\}$ or $\{z, z+1\}$ are bad, let $\varepsilon = \pm 1$ be such that $\{z, z+\varepsilon\}$ is bad and $\sigma(z+\varepsilon)$ is largest.
- Repeat for as long as $\{z, z+\varepsilon\}$ is bad:
  1. Switch z and $z+\varepsilon$ (in the cycle form of σ).
  2. If the elements preceding the last switched entries have consecutive values, switch them. Repeat 2.
  3. $z :=$ new rightmost entry of the cycle.

$$\pi = (11, 4, 10, 1, 7, 16, 9, 3, 5, 12, 20, 2, 6, 14, 18, 8, 13, 19, 15, 17, 21)$$
$$\sigma = (11, 4, 9, 3, 5)(16, 10, 1, 7, 14)(20, 2, 6, 13, 18, 8, 12, 19, 15, 17)$$

$z := 14$. $\varepsilon := 1$.

Switch 14 and 15.

The bijection $\varphi : \mathcal{C}_{n+1} \to \mathcal{S}_n$. Fixing bad pairs

For each but the last cycle of σ, from left to right:

- $z :=$ rightmost entry of the cycle.
  If $\{z, z-1\}$ or $\{z, z+1\}$ are bad, let $\varepsilon = \pm 1$ be such that $\{z, z+\varepsilon\}$ is bad and $\sigma(z+\varepsilon)$ is largest.
- Repeat for as long as $\{z, z+\varepsilon\}$ is bad:
  1. Switch z and $z+\varepsilon$ (in the cycle form of σ).
  2. If the elements preceding the last switched entries have consecutive values, switch them. Repeat 2.
  3. $z :=$ new rightmost entry of the cycle.

$$\pi = (11, 4, 10, 1, 7, 16, 9, 3, 5, 12, 20, 2, 6, 14, 18, 8, 13, 19, 15, 17, 21)$$
$$\sigma = (11, 4, 9, 3, 5)(16, 10, 1, 7, 15)(20, 2, 6, 13, 18, 8, 12, 19, 14, 17)$$

$z := 14$. $\varepsilon := 1$.

Switch 14 and 15.

The bijection $\varphi : \mathcal{C}_{n+1} \to \mathcal{S}_n$. Fixing bad pairs

For each but the last cycle of σ, from left to right:

- $z :=$ rightmost entry of the cycle.
  If $\{z, z-1\}$ or $\{z, z+1\}$ are bad, let $\varepsilon = \pm 1$ be such that $\{z, z+\varepsilon\}$ is bad and $\sigma(z+\varepsilon)$ is largest.
- Repeat for as long as $\{z, z+\varepsilon\}$ is bad:
  1. Switch z and $z+\varepsilon$ (in the cycle form of σ).
  2. If the elements preceding the last switched entries have consecutive values, switch them. Repeat 2.
  3. $z :=$ new rightmost entry of the cycle.

$$\pi = (11, 4, 10, 1, 7, 16, 9, 3, 5, 12, 20, 2, 6, 14, 18, 8, 13, 19, 15, 17, 21)$$
$$\sigma = (11, 4, 9, 3, 5)(16, 10, 1, \underline{7}, 15)(20, 2, 6, 13, 18, 8, 12, \underline{19}, 14, 17)$$

$z := 14$. $\varepsilon := 1$.

Switch 14 and 15.

The bijection $\varphi : \mathcal{C}_{n+1} \to \mathcal{S}_n$. Fixing bad pairs

For each but the last cycle of σ, from left to right:

- $z :=$ rightmost entry of the cycle.
  If $\{z, z-1\}$ or $\{z, z+1\}$ are bad, let $\varepsilon = \pm 1$ be such that $\{z, z+\varepsilon\}$ is bad and $\sigma(z+\varepsilon)$ is largest.
- Repeat for as long as $\{z, z+\varepsilon\}$ is bad:
  1. Switch z and $z+\varepsilon$ (in the cycle form of σ).
  2. If the elements preceding the last switched entries have consecutive values, switch them. Repeat 2.
  3. $z :=$ new rightmost entry of the cycle.

$$\pi = (11, 4, 10, 1, 7, 16, 9, 3, 5, 12, 20, 2, 6, 14, 18, 8, 13, 19, 15, 17, 21)$$
$$\sigma = (11, 4, 9, 3, 5)(16, 10, 1, 7, 15)(20, 2, 6, 13, 18, 8, 12, 19, 14, 17)$$

$z := 15$. $\varepsilon := 1$.

The bijection $\varphi : \mathcal{C}_{n+1} \to \mathcal{S}_n$. Fixing bad pairs

For each but the last cycle of σ, from left to right:

- $z :=$ rightmost entry of the cycle.
  If $\{z, z-1\}$ or $\{z, z+1\}$ are bad, let $\varepsilon = \pm 1$ be such that $\{z, z+\varepsilon\}$ is bad and $\sigma(z+\varepsilon)$ is largest.
- Repeat for as long as $\{z, z+\varepsilon\}$ is bad:
  1. Switch z and $z+\varepsilon$ (in the cycle form of σ).
  2. If the elements preceding the last switched entries have consecutive values, switch them. Repeat 2.
  3. $z :=$ new rightmost entry of the cycle.

$$\pi = (11, 4, 10, 1, 7, 16, 9, 3, 5, 12, 20, 2, 6, 14, 18, 8, 13, 19, 15, 17, 21)$$
$$\sigma = (11, 4, 9, 3, 5)(16, 10, 1, 7, 15)(20, 2, 6, 13, 18, 8, 12, 19, 14, 17)$$

$z := 15$. $\{15, 16\}$ is OK, so we are done.

The bijection $\varphi : \mathcal{C}_{n+1} \to \mathcal{S}_n$. Fixing bad pairs

For each but the last cycle of σ, from left to right:

- $z :=$ rightmost entry of the cycle.
  If $\{z, z-1\}$ or $\{z, z+1\}$ are bad, let $\varepsilon = \pm 1$ be such that $\{z, z+\varepsilon\}$ is bad and $\sigma(z+\varepsilon)$ is largest.
- Repeat for as long as $\{z, z+\varepsilon\}$ is bad:
  1. Switch z and $z+\varepsilon$ (in the cycle form of σ).
  2. If the elements preceding the last switched entries have consecutive values, switch them. Repeat 2.
  3. $z :=$ new rightmost entry of the cycle.

$$\begin{aligned}\pi &= (11, 4, 10, 1, 7, 16, 9, 3, 5, 12, 20, 2, 6, 14, 18, 8, 13, 19, 15, 17, 21)\\ \varphi(\pi) &= (11, 4, 9, 3, 5)(16, 10, 1, 7, 15)(20, 2, 6, 13, 18, 8, 12, 19, 14, 17)\end{aligned}$$

Define $\varphi(\pi) = \sigma$.

The descent sets are preserved

$$\begin{aligned}\pi &= (11,4,10,1,7,16,9,3,5,12,20,2,6,14,18,8,13,19,15,17,21)\\ \varphi(\pi) &= (11,4,9,3,5)(16,10,1,7,15)(20,2,6,13,18,8,12,19,14,17)\end{aligned}$$

In one-line notation,

$$\begin{aligned}\pi &= 7\cdot 6\cdot 5\ 10\ 12\ 14\ 16\cdot 13\cdot 3\cdot 1\ 4\ 20\cdot 19\cdot 18\cdot 16\cdot\ 9\ \ 21\cdot 8\ 15\cdot 2\ 11\\ \varphi(\pi) &= 7\cdot 6\cdot 5\ \ 9\ \ 11\ 13\ 15\cdot 12\cdot 3\cdot 1\ 4\ 19\cdot 18\cdot 17\cdot 16\cdot 10\ 20\cdot 8\ 14\cdot 2\end{aligned}$$

The inverse map $\varphi^{-1} : \mathcal{S}_n \longrightarrow \mathcal{C}_{n+1}$. First step

Given $\sigma \in \mathcal{S}_n$, write it in cycle form with the largest element of each cycle first, ordering the cycles by increasing first element:

$$\sigma = (11, 4, 9, 3, 5)(16, 10, 1, 7, 15)(20, 2, 6, 13, 18, 8, 12, 19, 14, 17) \in \mathcal{S}_{20}.$$

The inverse map $\varphi^{-1} : \mathcal{S}_n \longrightarrow \mathcal{C}_{n+1}$. First step

Given $\sigma \in \mathcal{S}_n$, write it in cycle form with the largest element of each cycle first, ordering the cycles by increasing first element:

$$\sigma = (11, 4, 9, 3, 5)(16, 10, 1, 7, 15)(20, 2, 6, 13, 18, 8, 12, 19, 14, 17) \in \mathcal{S}_{20}.$$

Remove parentheses and append $n + 1$:

$$\pi = (11, 4, 9, 3, 5\ ,\ 16, 10, 1, 7, 15\ ,\ 20, 2, 6, 13, 18, 8, 12, 19, 14, 17\ ,\ 21) \in \mathcal{C}_{21}.$$

The inverse map $\varphi^{-1} : \mathcal{S}_n \longrightarrow \mathcal{C}_{n+1}$. First step

Given $\sigma \in \mathcal{S}_n$, write it in cycle form with the largest element of each cycle first, ordering the cycles by increasing first element:

$$\sigma = (11, 4, 9, 3, 5)(16, 10, 1, 7, 15)(20, 2, 6, 13, 18, 8, 12, 19, 14, 17) \in \mathcal{S}_{20}.$$

Remove parentheses and append $n + 1$:

$$\pi = (11, 4, 9, 3, 5\ ,\ 16, 10, 1, 7, 15\ ,\ 20, 2, 6, 13, 18, 8, 12, 19, 14, 17\ ,\ 21) \in \mathcal{C}_{21}.$$

A pair $\{i, i + 1\}$ is *bad* if $\pi(i) > \pi(i + 1)$ but $\sigma(i) < \sigma(i + 1)$, or viceversa. We will fix the bad pairs.

The inverse map $\varphi^{-1} : \mathcal{S}_n \longrightarrow \mathcal{C}_{n+1}$. First step

Given $\sigma \in \mathcal{S}_n$, write it in cycle form with the largest element of each cycle first, ordering the cycles by increasing first element:

$$\sigma = (11, 4, 9, 3, 5)(16, 10, 1, 7, 15)(20, 2, 6, 13, 18, 8, 12, 19, 14, 17) \in \mathcal{S}_{20}.$$

Remove parentheses and append $n+1$:

$$\pi = (11, 4, 9, 3, 5\ ,\ 16, 10, 1, 7, 15\ ,\ 20, 2, 6, 13, 18, 8, 12, 19, 14, 17\ ,\ 21) \in \mathcal{C}_{21}.$$

A pair $\{i, i+1\}$ is *bad* if $\pi(i) > \pi(i+1)$ but $\sigma(i) < \sigma(i+1)$, or viceversa. We will fix the bad pairs.

Call *blocks* the pieces of π between removed parentheses.

The inverse map $\varphi^{-1} : \mathcal{S}_n \longrightarrow \mathcal{C}_{n+1}$. Fixing bad pairs

For each but the last block of π, from right to left:

- $z :=$ rightmost entry of the block.
  If $\{z, z-1\}$ or $\{z, z+1\}$ are bad, let $\varepsilon = \pm 1$ be such that $\{z, z+\varepsilon\}$ is bad and $\sigma(z+\varepsilon)$ is **smallest**.
- Repeat for as long as $\{z, z+\varepsilon\}$ is bad:
  1. Switch z and $z+\varepsilon$ (in the cycle form of π).
  2. If the elements preceding the last switched entries have consecutive values, switch them. Repeat 2.
  3. $z :=$ new rightmost entry of the block.

We obtain $\pi = \varphi^{-1}(\sigma)$.

Necklaces

$X = \{x_1, x_2, \dots\}_<$ linearly ordered alphabet.

A *necklace* of length ℓ is a circular arrangement of ℓ beads labeled with elements of X, up to cyclic rotation.

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- its *type* is the partition whose parts are the lengths of the necklaces;
- its *evaluation* is the monomial $x_1^{e_1} x_2^{e_2} \dots$ where e_i is the number of beads with label x_i.

Permutations and necklaces

Theorem (Gessel, Reutenauer '93)

$$|\{\pi \in \mathcal{S}_n \text{ with cycle structure } \lambda \text{ and descent composition } C\}| = \langle S_C, L_\lambda \rangle,$$

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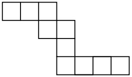

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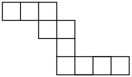

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Corollary (Gessel, Reutenauer '93)

Let $I = \{i_1, i_2, \ldots, i_k\}_< \subseteq [n-1]$, $\lambda \vdash n$. *Then*

$|\{\pi \in \mathcal{S}_n$ *with cycle structure* λ *and* $D(\pi) \subseteq I\}| =$

$|\{$*multisets of necklaces of type* λ *and evaluation* $x_1^{i_1} x_2^{i_2-i_1} \ldots x_k^{i_k-i_{k-1}} x_{k+1}^{n-i_k}\}|$.

Non-bijective proof using Gessel-Reutenauer

Goal : $|\{\pi \in \mathcal{C}_{n+1} \,:\, D(\pi) \cap [n-1] = I\}| = |\{\sigma \in \mathcal{S}_n \,:\, D(\sigma) = I\}|.$

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for all $I \subseteq [n-1]$. The statement follows by inclusion-exclusion.

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Let $\mathcal{T}_n$ be the set of n-cycles in one-line notation in which one entry has been replaced with 0.

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Corollary

For every n there is a bijection between $\mathcal{T}_n$ and $\mathcal{S}_n$ preserving the descent set.

Example:

$\mathcal{S}_3$	123	13·2	2·13	23·1	3·12	3·2·1
$\mathcal{T}_3$	012	03·1	3·02	23·0	3·02	3·1·0

THANK YOU