Expanding Hall-Littlewood polynomials in the dual Grothendieck basis

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Outline

Symmetric Functions

What is a symmetric function?

- Formal power series in $\mathbb{Q}[x_1, x_2, \ldots]$ which is
- invariant under permutation of indices.

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- invariant under permutation of indices.

Example

$$x_1^2 + x_2^2 + x_3^2 + \cdots \in \Lambda$$

$$x_1 + 2x_2 + x_3 + 2x_4 + \cdots \notin \Lambda$$

Symmetric Functions

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- Representation Theory
- Geometry/Topology
- Mathematical Physics

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- Representation Theory
- Geometry/Topology
- Mathematical Physics
- Beautiful Mathematics

The monomial basis

The *monomial symmetric functions* are indexed by partitions

$$\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k) \qquad \lambda_i \geq \lambda_{i+1}$$

$m_\lambda = \sum$ monomials whose exponent sequence is a rearrangement of λ

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$$m_\lambda = \sum \quad \text{monomials whose exponent sequence is a rearrangement of } \lambda$$

Example

$$\begin{aligned} m_{(2,1)} =& (x_1^2 x_2 + x_1 x_2^2) + (x_1 x_3^2 + x_1^2 x_3) + \ldots \\ &+ (x_2^2 x_3 + x_2 x_3^2) + \ldots \end{aligned}$$

The complete homogeneous basis

The *(complete) homogeneous symmetric function* is defined by

$$h_i = \sum \quad \text{all monomials of degree } i$$

$$h_\lambda = h_{\lambda_1} h_{\lambda_2} \dots h_{\lambda_k}$$

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Example

$$h_{(3)} = m_{(3)} + m_{(2,1)} + m_{(1,1,1)}$$
$$h_{(4,2,1)} = h_4 h_2 h_1$$

The Hall inner product

(Due to Philip Hall 1959) Defined by

$$\langle h_\lambda, m_\mu \rangle = \begin{cases} 1 & \text{if } \lambda = \mu \\ 0 & \text{otherwise} \end{cases}$$

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Proposition

If $\{f_\lambda\}$, $\{f_\lambda^*\}$ are dual bases and

$$f_\lambda = \sum_\mu M_{\lambda,\mu} m_\mu$$

then

$$h_\mu = \sum_\lambda M_{\lambda,\mu} f_\lambda^*$$

Semistandard Young tableaux

A left and bottom justified array of numbers, weakly increasing across rows and strictly increasing up columns.

Example

7	7			
5	6	6	8	
3	3	4	7	
1	1	2	2	2

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Shape: $(5, 4, 4, 2)$

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Shape: $(5, 4, 4, 2)$
Weight: $(2, 3, 2, 1, 1, 2, 3, 1)$

The Schur basis

Definition

The *Schur functions* are given by

$$s_\lambda = \sum_{T \in SSYT(\lambda)} x^{weight(T)}$$

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Example

$s_{(2,1)} =$	$x_1^2 x_2 +$	$x_1 x_2^2 +$	$2x_1x_2x_3 +$	$\cdots$
	2 / 1 1	2 / 1 2	3 / 1 2 ; 2 / 1 3	$\cdots$

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Example

$$s_{(2,1)} = \quad x_1^2 x_2 + \quad x_1 x_2^2 + \quad 2x_1x_2x_3 + \quad \cdots$$

$$\begin{array}{|c|c}\hline 2 & \\ \hline 1 & \multicolumn{1}{c|}{1} \\ \hline \end{array} \qquad \begin{array}{|c|c}\hline 2 & \\ \hline 1 & \multicolumn{1}{c|}{2} \\ \hline \end{array} \qquad \begin{array}{|c|c}\hline 3 & \\ \hline 1 & \multicolumn{1}{c|}{2} \\ \hline \end{array} \quad \begin{array}{|c|c}\hline 2 & \\ \hline 1 & \multicolumn{1}{c|}{3} \\ \hline \end{array} \qquad \cdots$$

Fact: Schur functions are a self-dual basis of the symmetric functions.

The Kostka numbers

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The *Kostka numbers* are defined by

$$K_{\lambda,\mu} = \text{\# of SSYT}(\lambda, \mu)$$

It follows that

$$s_\lambda = \sum_\mu K_{\lambda,\mu} m_\mu$$

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The matrix $K_{\lambda,\mu}$ is non-negative, integral and uni-triangular.

Interpretations as

- Irreducible characters of GL_n
- Irreducible characters of S_n
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among others.

Hall-Littlewood functions

The *Hall-Littlewood functions* form a basis for the symmetric functions with coefficient ring $\mathbb{Q}[t]$. Interpretations as

- Deformation of Weyl character formula
- Graded S_n character of certain cohomology rings
- Representation theory of matrices over finite fields

among others.

Hall-Littlewood functions

Fact: Hall-Littlewood functions interpolate between the Schur and homogeneous symmetric functions.

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Example

$$H_{(1,1,1)}[X;t] = s_{(1,1,1)} + (t^2 + t)s_{(2,1)} + t^3 s_{(3)}$$
$$H_{(1,1,1)}[X;0] = s_{(1,1,1)}$$
$$H_{(1,1,1)}[X;1] = s_{(1,1,1)} + 2s_{(2,1)} + s_{(3)} = h_{(1,1,1)}$$

Hall-Littlewood functions

The fact that $H_\mu[X;1] = h_\mu$ together with the fact

$$h_\mu = \sum_\lambda K_{\lambda,\mu} s_\lambda$$

implies

$$H_\mu[X;t] = \sum_\lambda K_{\lambda,\mu}(t) s_\lambda$$

where $K_{\lambda,\mu}(1) = K_{\lambda,\mu}$. This led Foulkes to conjecture

$$K_{\lambda,\mu}(t) = \sum_{T \in SSYT(\lambda,\mu)} t^{stat(T)}$$

Hall-Littlewood functions

Theorem (Lascoux-Schützenberger)

$$K_{\lambda,\mu}(t) = \sum_{T \in SSYT(\lambda,\mu)} t^{charge(T)}$$

where charge is a non-negative integer statistic defined on tableaux.

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where charge is a non-negative integer statistic defined on tableaux.

Example

$$h_{(1,1,1)} = s_{(1,1,1)} + 2s_{(2,1)} + s_{(3)}$$

$$\begin{array}{|c|}\hline 3 \\ \hline 2 \\ \hline 1 \\ \hline \end{array} \qquad \begin{array}{|c|c}\hline 3 & \multicolumn{1}{c}{} \\ \hline 1 & \multicolumn{1}{c|}{2} \\ \hline \end{array} \quad \begin{array}{|c|c}\hline 2 & \multicolumn{1}{c}{} \\ \hline 1 & \multicolumn{1}{c|}{3} \\ \hline \end{array} \qquad \begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline \end{array}$$

$$H_{(1,1,1)}[X;t] = t^0 s_{(1,1,1)} + (t^2 + t)s_{(2,1)} + t^3 s_{(3)}$$

The charge statistic

Example

3	4		
2	2	3	
1	1	1	2

3 4 2 2 3 1 1 1 2

The charge statistic

Example

3	4		
2	2	3	
1	1	1	2

							0	
3	4	2	2	3	1	1	1	2

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$charge(T) = 3$

Grothendieck functions

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- Introduced by Lascoux-Schützenberger (1982)
- Represent K-theory classes of structure sheaves of Schubert varieties
- Not homogeneous.
- Sign alternating by degree

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Example

$$G_{(1)} = s_{(1)} - s_{(1,1)} + s_{(1,1,1)} - s_{(1,1,1,1)} + \dots$$

Grothendieck functions

Theorem (Buch 2002)

$$G_\lambda = \sum L_{\lambda,\mu}(-1)^{|\mu|-|\lambda|} m_\mu$$

*where $L_{\lambda,\mu}$ is the number of **set-valued tableaux** of shape λ and weight μ.*

Set-valued tableaux

Example

234	45	
1	12	3

Set-valued tableaux

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$shape(T) = (3, 2)$
$weight(T) = (2, 2, 2, 2, 1)$

Set-valued tableaux

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$shape(T) = (3, 2)$
$weight(T) = (2, 2, 2, 2, 1)$

From

$$G_\lambda = \sum_\mu L_{\lambda,\mu} (-1)^{|\mu|-|\lambda|} m_\mu$$

we see

$$G_\lambda = s_\lambda \pm \text{ higher degree terms}$$

Dual Grothendieck functions

The expansion

$$G_\lambda = \sum_\mu L_{\lambda,\mu}(-1)^{|\mu|-|\lambda|} m_\mu$$

gives an implicit expansion of the dual basis by

$$h_\mu = \sum_\lambda L_{\lambda,\mu}(-1)^{|\mu|-|\lambda|} g_\lambda$$

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gives an implicit expansion of the dual basis by

$$h_\mu = \sum_\lambda L_{\lambda,\mu}(-1)^{|\mu|-|\lambda|} g_\lambda$$

Example

$$g_{(2,1)} = s_{(2,1)} + s_{(2)}$$

Fact: g_λ is a positive, finite sum of Schur functions.

Hall-Littlewoods and dual Grothendieck functions

The expansion

$$h_\mu = \sum_\lambda L_{\lambda,\mu}(-1)^{|\mu|-|\lambda|} g_\lambda$$

raises the question: Is the matrix $L_{\lambda,\mu}(t)$ defined by

$$H_\mu[X;t] = \sum_\lambda L_{\lambda,\mu}(t)(-1)^{|\mu|-|\lambda|} g_\lambda$$

nice?

Hall-Littlewoods and dual Grothendieck functions

Theorem (B-M)

$$H_\mu[X;t] = \sum_\lambda L_{\lambda,\mu}(t)(-1)^{|\mu|-|\lambda|} g_\lambda$$

where

$$L_{\lambda,\mu}(t) = \sum_{T \in SVT(\lambda,\mu)} t^{charge(T)}.$$

Compare with

$$H_\mu[X;t] = \sum_\lambda K_{\lambda,\mu}(t) s_\lambda$$

where

$$K_{\lambda,\mu}(t) = \sum_{T \in SSYT(\lambda,\mu)} t^{charge(T)}.$$

Hall-Littlewoods and dual Grothendieck functions

Example

$h_{(3,2)} =$

$$g_{(5)} \quad + g_{(4,1)} \quad + g_{(3,2)} \quad - g_{(3,1)} \quad - g_{(4)}$$

$$\begin{array}{|c|c|c|c|c|}\hline 1 & 1 & 1 & 2 & 2 \\ \hline \end{array} \qquad \begin{array}{|c|c|c|c|}\cline{1-1} 2 \\ \hline 1 & 1 & 1 & 2 \\ \hline \end{array} \qquad \begin{array}{|c|c|c|}\cline{1-2} 2 & 2 \\ \hline 1 & 1 & 1 \\ \hline \end{array} \qquad \begin{array}{|c|c|c|}\cline{1-1} 2 \\ \hline 1 & 1 & 12 \\ \hline \end{array} \qquad \begin{array}{|c|c|c|c|}\hline 1 & 1 & 12 & 2 \\ \hline \end{array}$$

$H_{(3,2)}[X;t] =$

$$t^2 g_{(5)} \quad + t^1 g_{(4,1)} \quad + t^0 g_{(3,2)} \quad - t^0 g_{(3,1)} \quad - t^1 g_{(4)}$$

Charge and set-valued tableaux

The definition of charge is almost identical to the classical case.

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1	12	3

4 3 2 5 4 1 2 1 3

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$charge(T) = 3$

Dual Grothendieck functions

Lam and Pylyavskyy gave a monomial expansion of the g_λ.

$$g_\lambda = \sum_\mu M_{\lambda,\mu} m_\mu$$

$M_{\lambda,\mu}$ is the number of *reverse plane partitions* of shape λ and "weight" μ.

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$$g_\lambda = \sum_{\mu} M_{\lambda,\mu} m_\mu$$

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Example

2	3		
2	2	4	
1	2	2	
1	1	1	1

$shape(T) = (4,3,3,2)$
$weight(T) = (4,3,1,1)$

To show

$$H_\mu[X;t] = \sum_\lambda L_{\lambda,\mu}(t)(-1)^{|\mu|-|\lambda|} g_\lambda$$

we show

$$H_\mu[X;t] = \sum_\lambda L_{\lambda,\mu}(t)(-1)^{|\mu|-|\lambda|} \sum_\nu M_{\lambda,\nu} m_\nu$$

by a sign-reversing involution $(S,T) \leftrightarrow (S,T)$
S: SVT of shape λ and weight μ
T: RPP of shape λ and weight ν
$sgn(S,T)$: $(-1)^{|\mu|-|\lambda|}$
Fixed points: $|\mu| = |\lambda| = |\nu|$

Final Thoughts

g_λ and $H_\mu[X;t]$ do not seem to "live in the same world", yet the combinatorics relating them is very natural.

Why?