

Expanding Hall-Littlewood polynomials in the dual Grothendieck basis

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October 23, 2009

Outline

- 1 Classical symmetric function theory
- 2 Hall-Littlewood polynomials
- 3 Grothendieck functions

Symmetric Functions

What is a symmetric function?

- Formal power series in $\mathbb{Q}[x_1, x_2, \dots]$ which is
- invariant under permutation of indices.

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Example

$$x_1^2 + x_2^2 + x_3^2 + \dots \in \Lambda$$

$$x_1 + 2x_2 + x_3 + 2x_4 + \dots \notin \Lambda$$

Symmetric Functions

What are they good for?

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- Representation Theory
- Geometry/Topology
- Mathematical Physics

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- Representation Theory
- Geometry/Topology
- Mathematical Physics
- Beautiful Mathematics

The monomial basis

The *monomial symmetric functions* are indexed by partitions

$$\lambda = (\lambda_1, \lambda_2, \dots, \lambda_k) \quad \lambda_i \geq \lambda_{i+1}$$

$m_\lambda = \sum$ monomials whose exponent sequence is
a rearrangement of λ

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Example

$$\begin{aligned} m_{(2,1)} = & (x_1^2 x_2 + x_1 x_2^2) + (x_1 x_3^2 + x_1^2 x_3) + \dots \\ & + (x_2^2 x_3 + x_2 x_3^2) + \dots \end{aligned}$$

The complete homogeneous basis

The *(complete) homogeneous symmetric function* is defined by

$$h_i = \sum \text{all monomials of degree } i$$

$$h_\lambda = h_{\lambda_1} h_{\lambda_2} \dots h_{\lambda_k}$$

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Example

$$h_{(3)} = m_{(3)} + m_{(2,1)} + m_{(1,1,1)}$$

$$h_{(4,2,1)} = h_4 h_2 h_1$$

The Hall inner product

(Due to Philip Hall 1959) Defined by

$$\langle h_\lambda, m_\mu \rangle = \begin{cases} 1 & \text{if } \lambda = \mu \\ 0 & \text{otherwise} \end{cases}$$

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Proposition

If $\{f_\lambda\}, \{f_\lambda^*\}$ are dual bases and

$$f_\lambda = \sum_{\mu} M_{\lambda,\mu} m_\mu$$

then

$$h_\mu = \sum_{\lambda} M_{\lambda,\mu} f_\lambda^*$$

Semistandard Young tableaux

A left and bottom justified array of numbers, weakly increasing across rows and strictly increasing up columns.

Example

7	7				
5	6	6	8		
3	3	4	7		
1	1	2	2	2	

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Shape: $(5, 4, 4, 2)$

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5	6	6	8		
3	3	4	7		
1	1	2	2	2	

Shape: $(5, 4, 4, 2)$

Weight: $(2, 3, 2, 1, 1, 2, 3, 1)$

The Schur basis

Definition

The *Schur functions* are given by

$$s_\lambda = \sum_{T \in \text{SSYT}(\lambda)} x^{\text{weight}(T)}$$

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2	1	1		2	1	2		3	1	2	1	3		...
...														

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2	1	1		2	1	2		3	1	2	1	3		...
1	1	1		1	2	1	2	1	2	1	3	3		...

Fact: Schur functions are a self-dual basis of the symmetric functions.

The Kostka numbers

Definition

The *Kostka numbers* are defined by

$$K_{\lambda, \mu} = \# \text{ of SSYT}(\lambda, \mu)$$

It follows that

$$s_{\lambda} = \sum_{\mu} K_{\lambda, \mu} m_{\mu}$$

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The matrix $K_{\lambda,\mu}$ is non-negative, integral and uni-triangular.

Interpretations as

- Irreducible characters of GL_n
- Irreducible characters of S_n
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among others.

Hall-Littlewood functions

The *Hall-Littlewood functions* form a basis for the symmetric functions with coefficient ring $\mathbb{Q}[t]$. Interpretations as

- Deformation of Weyl character formula
- Graded S_n character of certain cohomology rings
- Representation theory of matrices over finite fields

among others.

Hall-Littlewood functions

Fact: Hall-Littlewood functions interpolate between the Schur and homogeneous symmetric functions.

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Example

$$H_{(1,1,1)}[X; t] = s_{(1,1,1)} + (t^2 + t)s_{(2,1)} + t^3 s_{(3)}$$

$$H_{(1,1,1)}[X; 0] = s_{(1,1,1)}$$

$$H_{(1,1,1)}[X; 1] = s_{(1,1,1)} + 2s_{(2,1)} + s_{(3)} = h_{(1,1,1)}$$

Hall-Littlewood functions

The fact that $H_\mu[X; 1] = h_\mu$ together with the fact

$$h_\mu = \sum_{\lambda} K_{\lambda, \mu} s_{\lambda}$$

implies

$$H_\mu[X; t] = \sum_{\lambda} K_{\lambda, \mu}(t) s_{\lambda}$$

where $K_{\lambda, \mu}(1) = K_{\lambda, \mu}$. This led Foulkes to conjecture

$$K_{\lambda, \mu}(t) = \sum_{T \in \text{SSYT}(\lambda, \mu)} t^{\text{stat}(T)}$$

Hall-Littlewood functions

Theorem (Lascoux-Schützenberger)

$$K_{\lambda,\mu}(t) = \sum_{T \in \text{SSYT}(\lambda,\mu)} t^{\text{charge}(T)}$$

where charge is a non-negative integer statistic defined on tableaux.

Hall-Littlewood functions

Theorem (Lascoux-Schützenberger)

$$K_{\lambda,\mu}(t) = \sum_{T \in \text{SSYT}(\lambda,\mu)} t^{\text{charge}(T)}$$

where *charge* is a non-negative integer statistic defined on tableaux.

Example

$$h_{(1,1,1)} = s_{(1,1,1)} + 2s_{(2,1)} + s_{(3)}$$

3
2
1

3		2	
1	2	1	3

1	2	3
---	---	---

$$H_{(1,1,1)}[X; t] = t^0 s_{(1,1,1)} + (t^2 + t) s_{(2,1)} + t^3 s_{(3)}$$

The charge statistic

Example

3	4		
2	2	3	
1	1	1	2

3 4 2 2 3 1 1 1 2

The charge statistic

Example

3	4							
2	2	3						
1	1	1	2					

0

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0			0				0	
3	4	2	2	3	1	1	1	2

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3	4	2	2	3	1	1	1	2

$$\text{charge}(T) = 3$$

Grothendieck functions

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- Introduced by Lascoux-Schützenberger (1982)
- Represent K -theory classes of structure sheaves of Schubert varieties
- Not homogeneous.
- Sign alternating by degree

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Example

$$G_{(1)} = s_{(1)} - s_{(1,1)} + s_{(1,1,1)} - s_{(1,1,1,1)} + \dots$$

Grothendieck functions

Theorem (Buch 2002)

$$G_\lambda = \sum L_{\lambda,\mu} (-1)^{|\mu|-|\lambda|} m_\mu$$

where $L_{\lambda,\mu}$ is the number of **set-valued tableaux** of shape λ and weight μ .

Set-valued tableaux

Example

234	45	
1	12	3

Set-valued tableaux

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234	45	
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$$\text{shape}(T) = (3, 2)$$

$$\text{weight}(T) = (2, 2, 2, 2, 1)$$

Set-valued tableaux

Example

234	45	
1	12	3

$$\text{shape}(T) = (3, 2)$$

$$\text{weight}(T) = (2, 2, 2, 2, 1)$$

From

$$G_\lambda = \sum_{\mu} L_{\lambda, \mu} (-1)^{|\mu| - |\lambda|} m_\mu$$

we see

$$G_\lambda = s_\lambda \pm \text{higher degree terms}$$

Dual Grothendieck functions

The expansion

$$G_\lambda = \sum_{\mu} L_{\lambda, \mu} (-1)^{|\mu| - |\lambda|} m_\mu$$

gives an implicit expansion of the dual basis by

$$h_\mu = \sum_{\lambda} L_{\lambda, \mu} (-1)^{|\mu| - |\lambda|} g_\lambda$$

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Example

$$g_{(2,1)} = s_{(2,1)} + s_{(2)}$$

Fact: g_λ is a positive, finite sum of Schur functions.

Hall-Littlewoods and dual Grothendieck functions

The expansion

$$h_\mu = \sum_{\lambda} L_{\lambda,\mu} (-1)^{|\mu|-|\lambda|} g_\lambda$$

raises the question: Is the matrix $L_{\lambda,\mu}(t)$ defined by

$$H_\mu[X; t] = \sum_{\lambda} L_{\lambda,\mu}(t) (-1)^{|\mu|-|\lambda|} g_\lambda$$

nice?

Hall-Littlewoods and dual Grothendieck functions

Theorem (B-M)

$$H_\mu[X; t] = \sum_{\lambda} L_{\lambda, \mu}(t) (-1)^{|\mu| - |\lambda|} g_{\lambda}$$

where

$$L_{\lambda, \mu}(t) = \sum_{T \in SVT(\lambda, \mu)} t^{\text{charge}(T)}.$$

Compare with

$$H_\mu[X; t] = \sum_{\lambda} K_{\lambda, \mu}(t) s_{\lambda}$$

where

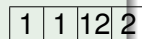
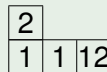
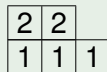
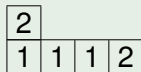
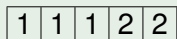
$$K_{\lambda, \mu}(t) = \sum_{T \in SSYT(\lambda, \mu)} t^{\text{charge}(T)}.$$

Hall-Littlewoods and dual Grothendieck functions

Example

$$h_{(3,2)} =$$

$$g_{(5)} \quad +g_{(4,1)} \quad +g_{(3,2)} \quad -g_{(3,1)} \quad -g_{(4)}$$



$$H_{(3,2)}[X; t] =$$

$$t^2 g_{(5)} \quad +t^1 g_{(4,1)} \quad +t^0 g_{(3,2)} \quad -t^0 g_{(3,1)} \quad -t^1 g_{(4)}$$

Charge and set-valued tableaux

The definition of charge is almost identical to the classical case.

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1	12	3

4 3 2 5 4 1 2 1 3

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Dual Grothendieck functions

Lam and Pylyavskyy gave a monomial expansion of the g_λ .

$$g_\lambda = \sum_{\mu} M_{\lambda, \mu} m_\mu$$

$M_{\lambda, \mu}$ is the number of *reverse plane partitions* of shape λ and “weight” μ .

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$$g_\lambda = \sum_{\mu} M_{\lambda, \mu} m_\mu$$

$M_{\lambda, \mu}$ is the number of *reverse plane partitions* of shape λ and “weight” μ .

Example

2	3		
2	2	4	
1	2	2	
1	1	1	1

$$\text{shape}(T) = (4, 3, 3, 2)$$

$$\text{weight}(T) = (4, 3, 1, 1)$$

To show

$$H_\mu[X; t] = \sum_{\lambda} L_{\lambda, \mu}(t) (-1)^{|\mu| - |\lambda|} g_\lambda$$

we show

$$H_\mu[X; t] = \sum_{\lambda} L_{\lambda, \mu}(t) (-1)^{|\mu| - |\lambda|} \sum_{\nu} M_{\lambda, \nu} m_\nu$$

by a sign-reversing involution $(S, T) \leftrightarrow (S, T)$

S : SVT of shape λ and weight μ

T : RPP of shape λ and weight ν

$\text{sgn}(S, T) : (-1)^{|\mu| - |\lambda|}$

Fixed points: $|\mu| = |\lambda| = |\nu|$

Final Thoughts

g_λ and $H_\mu[X; t]$ do not seem to “live in the same world”, yet the combinatorics relating them is very natural.

Why?