A Pieri rule for skew shapes

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Schur functions

A *partition* is a weakly decreasing sequence of nonnegative integers.

5				
4	7	8	9	
2	4	7	7	
1	1	3	4	7

$$T \in \mathrm{SSYT}(5,4,4,1)$$

A *semi-standard Young tableau of shape* λ is a filling of the cells of λ with positive integers such that entries weakly increase along rows and strictly increase up columns.

The *Schur function* $s_\lambda(X)$ is the generating function for $\mathrm{SSYT}(\lambda)$,

$$s_\lambda(X) = \sum_{T \in \mathrm{SSYT}(\lambda)} X^T \quad \text{where } X^T = x_1^{\#1's} x_2^{\#2's} \cdots$$

For $\mu \subset \lambda$, the *skew diagram* λ/μ is the set theoretic difference $\lambda - \mu$.

3				
1	3	3	7	
		1	4	
			2	5

$$T \in \mathrm{SSYT}((5,4,4,1)/(3,2))$$

A *semi-standard Young tableau of (skew) shape* ν is a filling of the cells of ν with positive integers such that entries weakly increase along rows and strictly increase up columns.

The *skew Schur function* $s_\nu(X)$ is the generating function for $\mathrm{SSYT}(\nu)$,

$$s_\nu(X) = \sum_{T \in \mathrm{SSYT}(\nu)} X^T, \qquad \text{where } X^T = x_1^{\#1's} x_2^{\#2's} \cdots$$

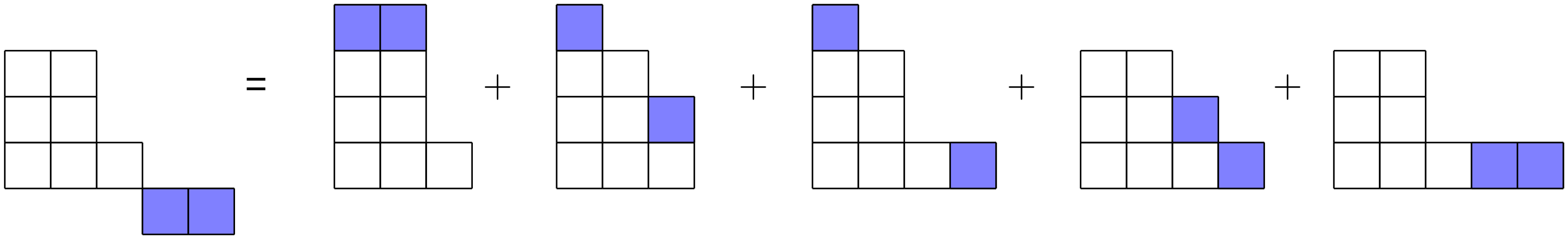
The *Pieri rule* gives a formula for expanding $s_\lambda s_{(n)}$ in terms of $\{s_\mu\}$.

Theorem. For λ a partition, we have

$$s_\lambda(X) s_{(n)}(X) = \sum_{\lambda^+/\lambda \ n-\text{hor. strip}} s_{\lambda^+}(X),$$

where λ^+/λ is a horizontal strip of size n.

Example: $s_{(3,2,2)} s_{(2)} = s_{(3,2,2,2)} + s_{(3,3,2,1)} + s_{(4,2,2,1)} + s_{(4,3,2)} + s_{(5,2,2)}$.

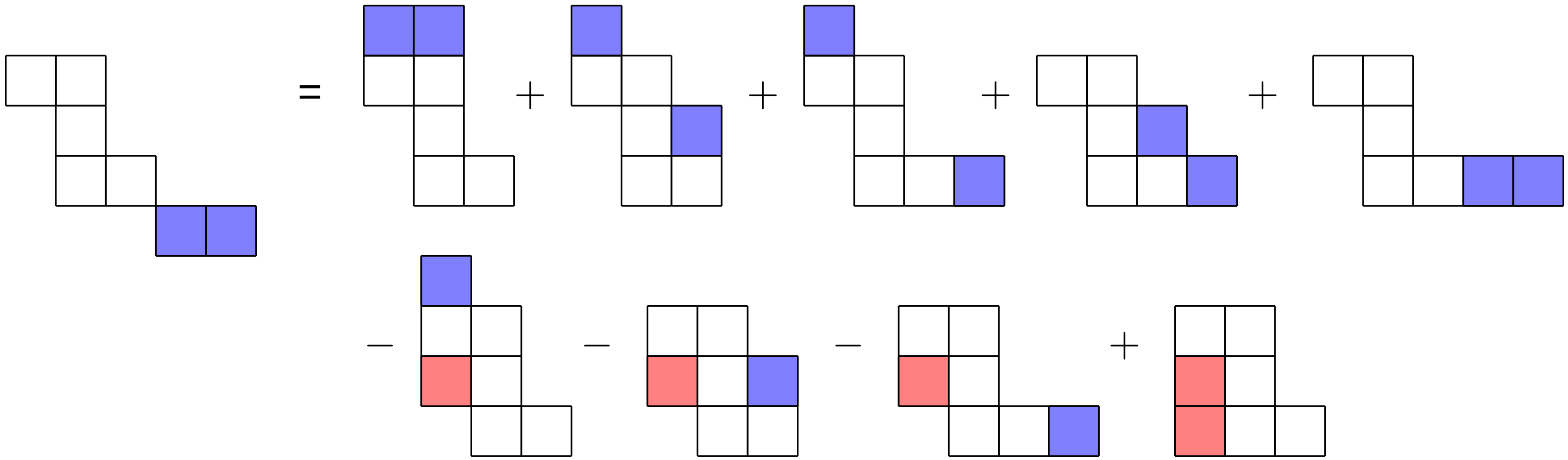


Theorem. (Assaf-McNamara 2009) For λ/μ a skew diagram, we have

$$s_{\lambda/\mu} s_{(n)} = \sum_{k=0}^{n} (-1)^k \sum_{\substack{\lambda^+/\lambda \ (n-k)\text{-hor. strip} \\ \mu/\mu^- \ k\text{-vert. strip}}} s_{\lambda^+/\mu^-} ,$$

with λ^+/λ a horizontal strip of size $n-k$, μ/μ^- a vertical strip of size k.

Example: $s_{322/11} s_2 = s_{3222/11} + s_{3321/11} + s_{4221/11} + s_{432/11} + s_{522/11}$
$- s_{3221/1} - s_{332/1} - s_{422/1} + s_{322}$.



Robinson-Schensted row insertion algorithm:

$$\begin{array}{|c|c|c|c|c|} \hline 5 \\ \cline{1-1} 4 & \mathbf{7} & 8 & 9 \\ \cline{1-4} 2 & \mathbf{4} & 7 & 7 \\ \hline 1 & 1 & \mathbf{3} & 4 & 7 \\ \hline \end{array} \leftarrow \mathbf{2} \quad = \quad \begin{array}{|c|c|c|c|c|} \hline 5 & \mathbf{7} \\ \hline 4 & \mathbf{4} & 8 & 9 \\ \cline{1-4} 2 & \mathbf{3} & 7 & 7 \\ \hline 1 & 1 & \mathbf{2} & 4 & 7 \\ \hline \end{array}$$

Proposition. For any $T \in \mathrm{SSYT}(\lambda)$ and any k, $T \leftarrow k \in \mathrm{SSYT}(\lambda \cup \square)$. Moreover, this process is reversible.

Lemma. Repeated row insertions $T \leftarrow k_1 \leftarrow k_2 \leftarrow \cdots \leftarrow k_n$ add a horizontal strip of size n if and only if $k_1 \leq k_2 \leq \cdots \leq k_n$.

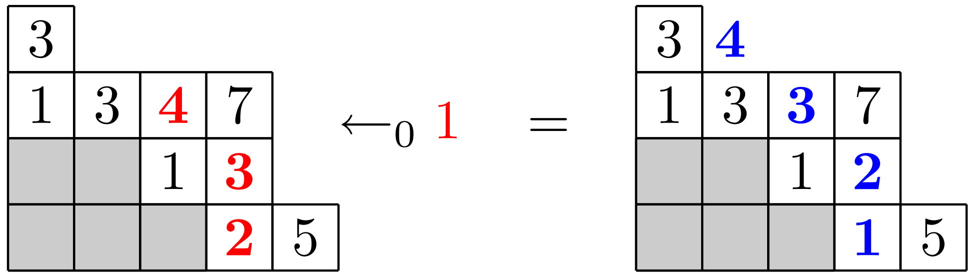
Theorem. The R-S row insertion algorithm gives a bijection

$$\mathrm{SSYT}(\lambda) \times \mathrm{SSYT}(n) \quad \stackrel{\sim}{\longleftrightarrow} \quad \bigsqcup_{\lambda^+/\lambda \ n-\text{hor. strip}} \mathrm{SSYT}(\lambda^+)$$

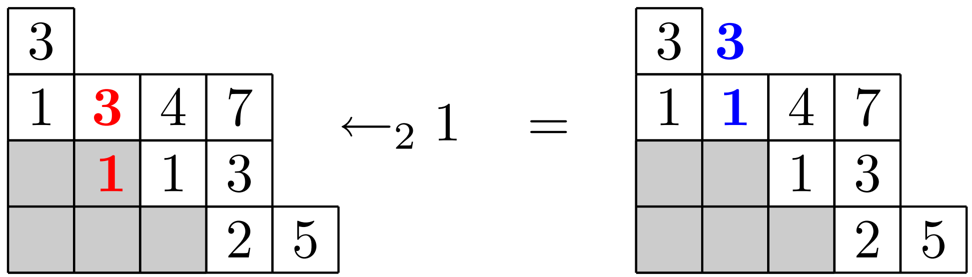
Row insertion for skew shapes

Slightly more general notion of R-S row insertion for skew shapes.

external insertion:

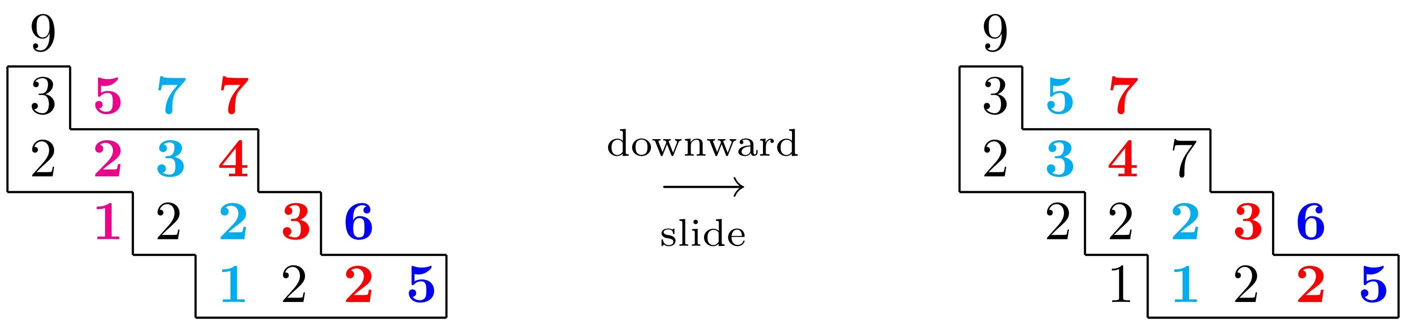


internal insertion:

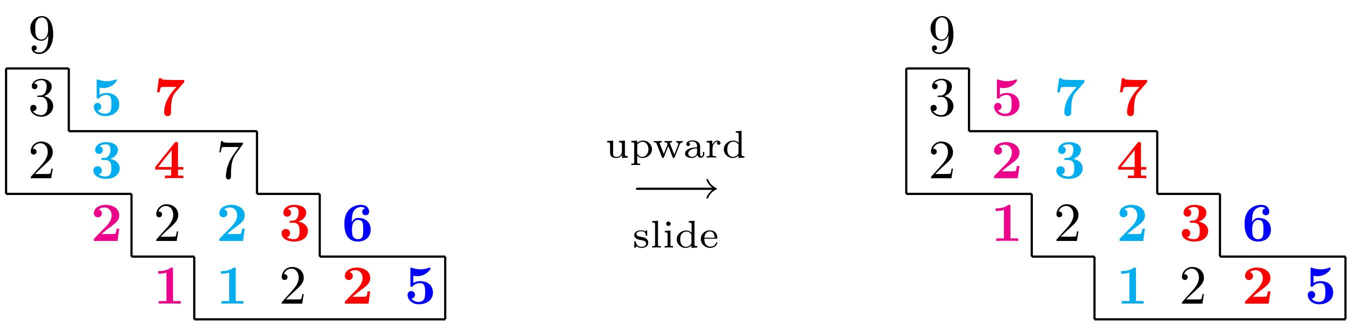


Proposition. External and internal row insertions are well-defined, reversible operations on SSYT of skew shape.

Construct a sign-reversing involution on $\mathrm{SSYT}(\lambda^+/\mu^-)$.



Proposition. A *downward slide* will remove a box of λ^+/λ and add a box to μ/μ^- with overall shape λ plus a horizontal strip minus μ minus a vertical strip. Moreover, this process is reversible.



Theorem. (Assaf-McNamara) Downward and upward slides establish a sign-reversing involution on

$$\bigsqcup_{k=0}^{n} \bigsqcup_{\substack{\lambda^+/\lambda \ (n-k)\text{-hor. strip} \\ \mu/\mu^- \ k\text{-vert. strip}}} \mathrm{SSYT}(\lambda^+/\mu^-)$$

such that **fixed points** are in bijection with $\mathrm{SSYT}(\lambda/\mu) \times \mathrm{SSYT}(n)$.

Theorem.(Assaf-McNamara) For λ/μ a skew diagram, we have

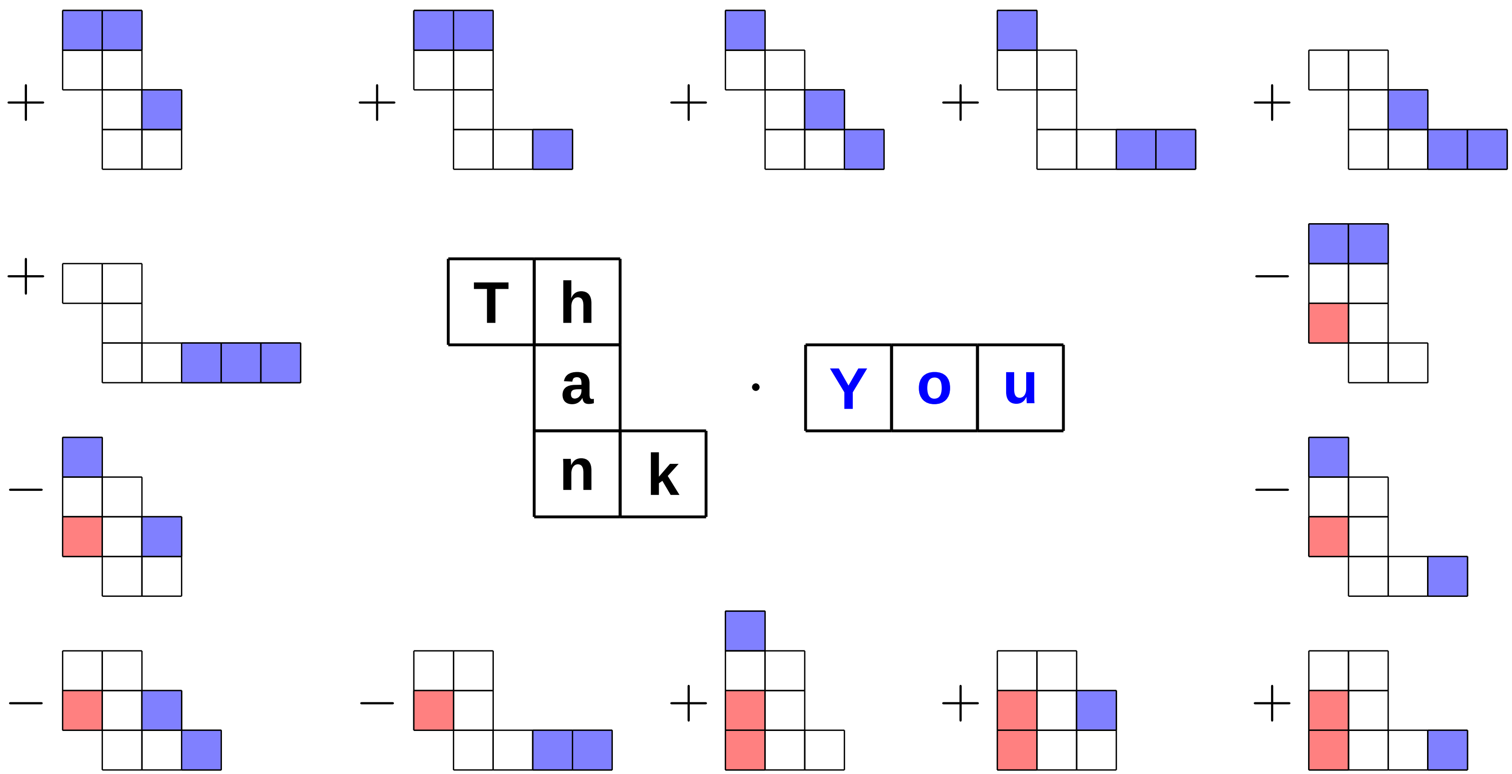
$$s_{\lambda/\mu} s_{(n)} = \sum_{k=0}^{n} (-1)^k \sum_{\substack{\lambda^+/\lambda \ (n-k)\text{-hor. strip} \\ \mu/\mu^- \ k\text{-vert. strip}}} s_{\lambda^+/\mu^-} ,$$

with λ^+/λ a horizontal strip of size $n-k$, μ/μ^- a vertical strip of size k.

Also a symmetric function proof by Thomas Lam based on the $n = 1$ proof of Richard Stanley, using the Hall inner product.

Preprint available on the arXiv:

Sami H. Assaf and Peter R.W. McNamara. *A Pieri rule for skew shapes*. arXiv:0908.0345



A skew Littlewood-Richardson rule

Conjecture. (Assaf-McNamara) For λ/μ and σ/τ,

$$s_{\lambda/\mu} s_{\sigma/\tau} = \sum_{\substack{T^+ \in \mathrm{SSYT}(\lambda^+/\lambda) \\ T^- \in \mathrm{ASSYT}(\mu/\mu^-)}} (-1)^{|\mu/\mu^-|} s_{\lambda^+/\mu^-},$$

summing over SSYTs T^+ of shape λ^+/λ for some $\lambda^+ \supseteq \lambda$, and ASSYTs T^- of shape μ/μ^- for some $\mu^- \subseteq \mu$, such that

- the content of $T^- \cup T^+$ is the difference $\sigma - \tau$, and
- the reverse reading word of (T^-, T^+) is τ-Yamanouchi.

This conjecture was recently proven by Thomas Lam, Aaron Lauve and Frank Sottile in the general setting of Hopf Algebras:

T. Lam, A. Lauve and F. Sottile. *Skew Littlewood-Richardson rules from Hopf algebras*. arXiv:0908.3714