A Pieri rule for skew shapes

Sami H. Assaf[1]

Department of Mathematics, Massachusetts Institute of Technology, Cambridge, MA 02139, USA

Peter R. W. McNamara

Department of Mathematics, Bucknell University, Lewisburg, PA 17837, USA

Abstract

The Pieri rule expresses the product of a Schur function and a single row Schur function in terms of Schur functions. We extend the classical Pieri rule by expressing the product of a skew Schur function and a single row Schur function in terms of skew Schur functions. Like the classical rule, our rule involves simple additions of boxes to the original skew shape. Our proof is purely combinatorial and extends the combinatorial proof of the classical case.

Key words: Pieri rule, skew Schur functions, Robinson-Schensted
2000 MSC: 05E05, 05E10, 20C30

1. Introduction

The basis of Schur functions is arguably the most interesting and important basis for the ring of symmetric functions. This is due not just to their elegant combinatorial definition, but more broadly to their connections to other areas of mathematics. For example, they are intimately tied to the cohomology ring of the Grassmannian, and they appear in the representation theory of the symmetric group and of the general and special linear groups.

It is therefore natural to consider the expansion of the product $s_\lambda s_\mu$ of two Schur functions in the basis of Schur functions. The Littlewood–Richardson rule [11, 19, 24, 25], which now comes in many different forms ([22] is one starting point), allows us to determine this expansion. However, more basic than the Littlewood–Richardson rule is the Pieri rule, which gives a simple, beautiful and more intuitive answer for the special case when $\mu = (n)$, a partition of length 1. Though we will postpone the preliminary definitions to Section 2 and the statement of the Pieri rule to Section 3, stating the rule in a rough form will

Email addresses: `sassaf@math.mit.edu` (Sami H. Assaf), `peter.mcnamara@bucknell.edu` (Peter R. W. McNamara)

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give its flavor. For a partition λ and a positive integer n, the Pieri rule states that $s_\lambda s_n$ is a sum of Schur functions s_{λ^+}, where λ^+ is obtainable by adding cells to the diagram of λ according to a certain simple rule. The Pieri rule's prevalence is highlighted by its adaptions to many other settings, including Schubert polynomials [8, 10, 13, 20, 26], LLT polynomials [5], Hall–Littlewood polynomials [14], Jack polynomials [9, 21], and Macdonald polynomials [4, 12].

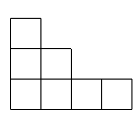
It is therefore surprising that there does not appear to be a known adaption of the Pieri rule to the most well-known generalization of Schur functions, namely skew Schur functions. We fill this gap in the literature with a natural extension of the Pieri rule to the skew setting. Reflecting the simplicity of the classical Pieri rule, the skew Pieri rule states that for a skew shape λ/μ and a positive integer n, $s_{\lambda/\mu}s_n$ is a signed sum of skew Schur functions s_{λ^+/μ^-}, where λ^+/μ^- is obtainable by adding cells to the diagram of λ/μ according to a certain simple rule. Our proof is purely combinatorial, using a sign-reversing involution that reflects the combinatorial proof of the classical Pieri rule. After reading an earlier version of this manuscript, which included an algebraic proof of the case $n = 1$ due to Richard Stanley, Thomas Lam provided a complete algebraic proof of our skew Pieri rule.

It is natural to ask if our skew Pieri rule can be extended to give a "skew" version of the Littlewood–Richardson rule, and we include such a rule as a conjecture in Section 6. This conjecture has been proved by Lam, Aaron Lauve and Frank Sottile in [7] using Hopf algebras. It remains an open problem to find a combinatorial proof of the skew Littlewood–Richardson rule.

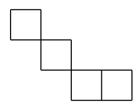
The remainder of this paper is organized as follows. In Section 2, we give the necessary symmetric function background. In Section 3, we state the classical Pieri rule and introduce our skew Pieri rule. In Section 4, we give a variation from [17] of the Robinson–Schensted–Knuth algorithm, along with relevant properties. This algorithm is then used in Section 5 to define our sign-reversing involution, which we then use to prove the skew Pieri rule. We conclude in Section 6 with two connections to the Littlewood–Richardson rule. Lam's algebraic proof appears in Appendix A.

2. Preliminaries

We follow the terminology and notation of [12, 22] for partitions and tableaux, except where specified. Letting $\mathbb{N}$ denote the nonnegative integers, a *partition* λ of $n \in \mathbb{N}$ is a weakly decreasing sequence $(\lambda_1, \lambda_2, \ldots \lambda_l)$ of positive integers whose sum is n. It will be convenient to set $\lambda_k = 0$ for $k > l$. We also let $\emptyset$ denote the unique partition with $l = 0$. We will identify λ with its *Young diagram* in "French notation": represent the partition λ by the unit square cells with top-right corners $(i, j) \in \mathbb{N} \times \mathbb{N}$ such that $1 \leq i \leq \lambda_j$. For example, the partition $(4, 2, 1)$, which we abbreviate as 421, has Young diagram

 .

Define the *conjugate* or *transpose* λ^t of λ to be the partition with λ_i cells in column i. For example, $421^t = 3211$. For another partition μ, we write $\mu \subseteq \lambda$ whenever μ is contained within λ (as Young diagrams); equivalently $\mu_i \leq \lambda_i$ for all i. In this case, we define the *skew shape* λ/μ to be the set theoretic difference $\lambda - \mu$. In particular, the partition λ is the skew shape $\lambda/\emptyset$. We call the number of cells of λ/μ its *size*, denoted $|\lambda/\mu|$. We say that a skew shape forms a *horizontal strip* (resp. *vertical strip*) if it contains no two cells in the same column (resp. row). A *k-horizontal strip* is a horizontal strip of size k, and similarly for vertical strips. For example, the skew shape $421/21$ is a 4-horizontal strip:

 .

With another skew shape σ/τ, we let $(\lambda/\mu) * (\sigma/\tau)$ denote the skew shape obtained by positioning λ/μ so that its bottom right cell is immediately above and left of the top left cell of σ/τ. For example, the horizontal strip $421/21$ above could alternatively be written as $(21/1) * (2)$ or as $(1) * (31/1)$.

A *Young tableau* of shape λ/μ is a map from the cells of λ/μ to the positive integers. A *semistandard Young tableau* (SSYT) is such a filling which is weakly increasing from left-to-right along each row and strictly increasing up each column, such as

$$\begin{array}{|c|c|c|c|} \hline 5 & \multicolumn{3}{c}{} \\ \hline 3 & 3 & 5 & \multicolumn{1}{c}{} \\ \hline \multicolumn{1}{c|}{} & 1 & 2 & 7 \\ \cline{2-4} \end{array} .$$

The *content* of an SSYT T is the sequence π such that T has π_i cells with entry i; in this case $\pi = (1, 1, 2, 0, 2, 0, 1)$.

We let Λ denote the ring of symmetric functions in the variables $x = (x_1, x_2, \ldots)$ over $\mathbb{Q}$, say. We will use three familiar bases from [12, 22] for Λ: the elementary symmetric functions e_λ, the complete homogeneous symmetric functions h_λ and, most importantly, the *Schur functions* s_λ. The Schur functions form an orthonormal basis for Λ with respect to the Hall inner product and may be defined in terms of SSYTs by

$$s_\lambda = \sum_{T \in \mathrm{SSYT}(\lambda)} x^T, \tag{2.1}$$

where the sum is over all SSYTs of shape λ and where x^T denotes the monomial $x_1^{\pi_1} x_2^{\pi_2} \cdots$ when T has content π. Replacing λ by λ/μ in (2.1) gives the definition of the *skew Schur function* $s_{\lambda/\mu}$, where the sum is now over all SSYTs of shape λ/μ. For example, the SSYT shown above contributes the monomial $x_1 x_2 x_3^2 x_5^2 x_7$ to $s_{431/1}$.

3. The skew Pieri rule

The celebrated Pieri rule gives an elegant method for expanding the product $s_\lambda s_n$ in the Schur basis. This rule was originally stated in [15] in the setting of Schubert Calculus. Recall that the single row Schur function s_n equals the complete homogeneous symmetric function h_n. Recall also the involution ω on Λ, which may be defined by sending the Schur function s_λ to s_{λ^t} or equivalently by sending h_k to e_k. Thus the Schur function s_{1^n} equals the elementary symmetric function e_n, where 1^n denotes a single column of size n.

Theorem 3.1 ([15]). *For any partition λ and positive integer n, we have*

$$s_\lambda s_n = s_\lambda h_n = \sum_{\lambda^+/\lambda \ n\text{-hor. strip}} s_{\lambda^+}, \tag{3.1}$$

where the sum is over all partitions λ^+ such that λ^+/λ is a horizontal strip of size n.

Applying the involution ω to (3.1), we get the dual version of the Pieri rule:

$$s_\lambda s_{1^n} = s_\lambda e_n = \sum_{\lambda^+/\lambda \ n\text{-vert. strip}} s_{\lambda^+} \ , \tag{3.2}$$

where the sum is now over all partitions λ^+ such that λ^+/λ is a vertical strip of size n.

A simple application of Theorem 3.1 gives

$$s_{322}s_2 = s_{3222} + s_{3321} + s_{4221} + s_{432} + s_{522},$$

as represented diagrammatically in Figure 1.

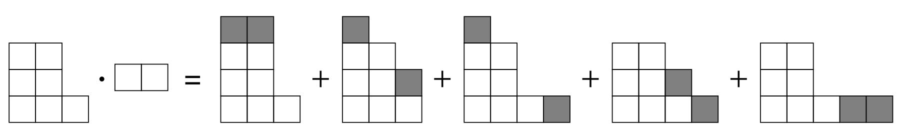


Figure 1: The expansion of $s_{322}s_2$ by the Pieri rule.

Given the simplicity of (3.1), it is natural to hope for a simple expression for $s_{\lambda/\mu}s_n$ in terms of skew Schur functions. This brings us to our main result.

Theorem 3.2. *For any skew shape λ/μ and positive integer n, we have*

$$s_{\lambda/\mu}s_n = s_{\lambda/\mu}h_n = \sum_{k=0}^{n}(-1)^k \sum_{\substack{\lambda^+/\lambda \ (n-k)\text{-hor. strip} \\ \mu/\mu^- \ k\text{-vert. strip}}} s_{\lambda^+/\mu^-} \ , \tag{3.3}$$

where the second sum is over all partitions λ^+ and μ^- such that λ^+/λ is a horizontal strip of size $n-k$ and μ/μ^- is a vertical strip of size k.

Observe that when $\mu = \emptyset$, Theorem 3.2 specializes to Theorem 3.1. Again, we can apply the ω transformation to obtain the dual version of the skew Pieri rule.

Corollary 3.3. *For any skew shape λ/μ and any positive integer k, we have*

$$s_{\lambda/\mu} s_{1^n} = s_{\lambda/\mu} e_n = \sum_{k=0}^{n} (-1)^k \sum_{\substack{\lambda^+/\lambda \ (n-k)\text{-vert. strip} \\ \mu/\mu^- \ k\text{-hor. strip}}} s_{\lambda^+/\mu^-},$$

where the sum is over all partitions λ^+ and μ^- such that λ^+/λ is a vertical strip of size $n-k$ and μ/μ^- is a horizontal strip of size k.

Example 3.4. A direct application of Theorem 3.2 gives

$$\begin{aligned} s_{322/11} s_2 = {} & s_{3222/11} + s_{3321/11} + s_{4221/11} + s_{432/11} + s_{522/11} \\ & - s_{3221/1} - s_{332/1} - s_{422/1} + s_{322}, \end{aligned}$$

as represented diagrammatically in Figure 2.

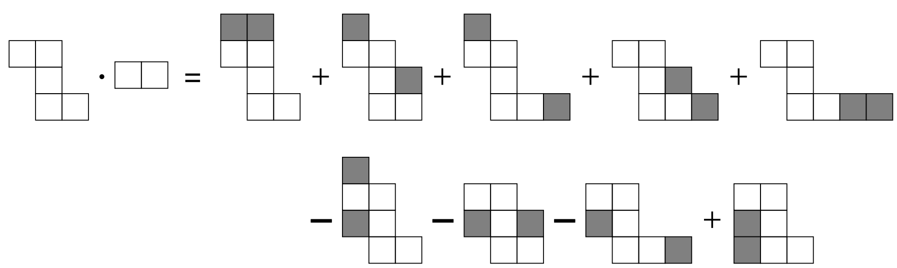


Figure 2: The expansion of $s_{322/11} s_2$ by the skew Pieri rule.

As (3.3) contains negative signs, our approach to proving Theorem 3.2 will be to construct a sign-reversing involution on SSYTs of shapes of the form λ^+/μ^-. We will then provide a bijection between SSYTs of shape λ^+/μ^- that are fixed under the involution and SSYTs of shape $(\lambda/\mu) * (n)$. The result then follows from the fact that $s_{(\lambda/\mu)*(n)} = s_{\lambda/\mu} s_n$.

4. Row insertion

In order to describe our sign-reversing involution, we will need the Robinson–Schensted–Knuth (RSK) row insertion algorithm on SSYTs [16, 18, 3]. For a thorough treatment of this algorithm along with many applications, we recommend [22]. In fact, we will use an analogue of the algorithm for SSYTs of *skew* shape from [17]. There, row insertion comes in two forms, external and internal row insertion. External row insertion, which we now define, is just like the classical RSK insertion.

Definition 4.1. Let T be an SSYT of arbitrary skew shape and choose a positive integer k. Define the *external row insertion of k into T*, denoted $T \leftarrow_0 k$, as follows: if k is weakly larger than all entries in row 1 of T, then add k to the right end of the row and terminate the process. Otherwise, find the leftmost cell in row 1 of T whose entry, say k', is greater than k. Replace this entry by k and then row insert k' into T at row 2 using the procedure just described. Repeat the process until some entry comes to rest at the right end of a row.

Example 4.2. Let $\lambda/\mu = 7541/32$ and $\lambda^+/\mu^- = 7542/31$ so that the outlined entries below are those in λ/μ. The result of externally row inserting a 2 is shown below, with changed cells circled.

$$\begin{array}{ccccccc} 4 & 5 & \bigcirc & & & & \\ 2 & 2 & 7 & 7 & & & \\ & 1 & 2 & 3 & 4 & & \\ & & & 2 & 2 & 3 & 6 \end{array} \quad \leftarrow_0 2 \quad = \quad \begin{array}{ccccccc} 4 & 5 & 7 & & & & \\ 2 & 2 & 4 & 7 & & & \\ & 1 & 2 & 3 & 3 & & \\ & & & 2 & 2 & 2 & 6 \end{array} \tag{4.1}$$

An *inside corner* (resp. *outside corner*) of an SSYT T is a cell that has no cell of T immediately below or to its left (resp. above or to its right). Therefore, inside and outside corners are those individual cells whose removal from T still yields an SSYT of skew shape.

Definition 4.3. Let T be an SSYT of arbitrary skew shape and let T have an inside corner in row r with entry k. Define the *internal row insertion of k from row r into T*, denoted $T \leftarrow_r k$, as the removal of k from row r and its insertion, using the rules for external row insertion, into row $r+1$. The insertion proceeds until some entry comes to rest at the right end of a row.

We could regard external insertions as internal insertions from row 0, explaining our notation. We will simply write $T \leftarrow k$ when specifying the type or row of the insertion is unnecessarily cumbersome.

Example 4.4. Taking T as the SSYT on the right in (4.1), the result of internally row inserting the 1 from row 2 is shown below.

$$\begin{array}{ccccccc} \bigcirc & & & & & & \\ 4 & 5 & 7 & & & & \\ 2 & 2 & 4 & 7 & & & \\ & 1 & 2 & 3 & 3 & & \\ & & & 2 & 2 & 2 & 6 \end{array} \quad \leftarrow_2 1 \quad = \quad \begin{array}{ccccccc} 4 & & & & & & \\ 2 & 5 & 7 & & & & \\ 1 & 2 & 4 & 7 & & & \\ & \bigcirc & 2 & 3 & 3 & & \\ & & & 2 & 2 & 2 & 6 \end{array} \tag{4.2}$$

For both types of insertion, we must be a little careful when inserting an entry into an empty row, say row i: in this case $\lambda_i = \mu_i$ and the entry must be placed in column $\lambda_i + 1$.

Note that an internal insertion results in the same multiset of entries while an external insertion adds an entry. It is not difficult to check that either operation results in an SSYT.

We will also need to invert row insertions, again for skew shapes and following [17].

Definition 4.5. Let T be an SSYT of arbitrary skew shape and choose an outside corner c of T, say with entry k. Define the *reverse row insertion of c from T*, denoted $T \to c$, by deleting c from T and reverse inserting k into the row below, say row r, as follows: if $r = 0$, then the procedure terminates. Otherwise, if k is weakly smaller than all entries in row r, then place k at the left end of row r and terminate the procedure. Otherwise, find the rightmost cell in row r whose entry, say k', is less than k. Replace this entry by k and then reverse row insert k' into row $r-1$ using the procedure just described.

Example 4.6. In (4.1), reverse row insertion of the cell containing the circled 7 from the SSYT T on the right results in the SSYT on the left, and similarly in (4.2) for the circled 4.

As with row insertion, it follows from the definition that the resulting array will again be an SSYT. Observe that the first type of termination mentioned in Definition 4.5 corresponds to reverse external row insertion, and we then say that k *lands in row* 0. The second type of termination corresponds to reverse internal row insertion, and we then say that k *lands in row* r. In both cases, we will call the entry k left at the end of the procedure the *final entry of $T \to c$*. The following lemma, which follows immediately from Definitions 4.1, 4.3 and 4.5, formalizes the bijectivity of row and reverse row insertion.

Lemma 4.7. *Let T be an SSYT of skew shape.*

a. *If S is the result of $T \leftarrow k$ for some positive integer k, then $S \to c$ results in T, where c is the unique non-empty cell of S that is empty in T.*
b. *If S is the result of $T \to c$ for some removable cell c of T and the final entry k of $T \to c$ lands in row $r \geq 0$, then $S \leftarrow_r k$ results in T.*

For both row insertion and reverse row insertion, we will often want to track the cells affected by the procedure. Therefore define the *bumping path of the row insertion $T \leftarrow k$* (resp. the *reverse bumping path of a reverse row insertion $T \to c$*) to be the set of cells in T, as well as those empty cells, where the entries differ from the corresponding entries in $T \leftarrow k$ (resp. $T \to c$). The cells of the bumping paths for row insertion and reverse row insertion are circled in (4.1) and (4.2). Note that the fact that the bumping path and reverse bumping path are equal in each of these examples is a consequence of Lemma 4.7.

It is easy to see that the bumping paths always move weakly right from top to bottom in the case of column-strict tableaux. The following bumping lemma will play a crucial role in defining our sign-reversing involution and in proving its relevant properties.

Lemma 4.8. *Let T be an SSYT of skew shape and let k, k' be positive integers. Let B be the bumping path of $T \leftarrow k$ and let B' be the bumping path of $(T \leftarrow k) \leftarrow k'$.*

a. *If B is strictly left of B' in any row r, then B is strictly left of B' in every row they both occupy. Moreover, the top cells of B and B' form a horizontal strip.*

b. *If both row insertions are external, then B is strictly left of B' in every row they both occupy if and only if $k \leq k'$.*

c. *Suppose C' is the reverse bumping path of $T \rightarrow c'$ with final entry k' and C is the reverse bumping path of $(T \rightarrow c') \rightarrow c$ with final entry k. If c is strictly left of c', then C is strictly left of C' in every row they both occupy. If, in addition, both reverse row insertions land in row 0, then $k \leq k'$.*

Proof. For general i, let B_i (resp. B'_i) be the cell of B (resp. B') in row i, say with entry b_i in T (resp. b'_i in $T \leftarrow k$).

(a) Suppose B_r is strictly left of B'_r for some r for which B_{r+1} and B'_{r+1} both exist. Then b_r will occupy the cell B_{r+1} in $T \leftarrow k$, and since $b'_r \geq b_r$, B'_{r+1} will be strictly right of B_{r+1}. To show that B_{r-1} is strictly left of B'_{r-1} assuming both exist, let c denote the cell that is empty in T but non-empty in $T \leftarrow k$ and let c' denote the cell that is empty in $T \leftarrow k$ but not in $(T \leftarrow k) \leftarrow k'$. By Lemma 4.7(a), B (resp. B') is also the reverse bumping path of $(((T \leftarrow k) \leftarrow k') \rightarrow c') \rightarrow c$ (resp. $((T \leftarrow k) \leftarrow k') \rightarrow c'$). We know that b_{r-1} will occupy the cell B_r in $(T \leftarrow k) \leftarrow k'$ while b'_{r-1} will occupy the cell B'_r, implying $b_{r-1} \leq b'_{r-1}$. As a result, considering the reverse row insertions of c' and then c, we deduce that B_{r-1} is strictly left of B'_{r-1}.

Now consider the top cells B_r and B'_s of B and B' respectively. Since B_r is the top cell of B, we know that it is at the right end of row r. When $s \geq r$, we know that B'_r is strictly right of B_r. But then B'_r is empty in $T \leftarrow k$ and so we must have $s = r$. We conclude that $s \leq r$ and the result follows.

(b) If $k \leq k'$, then B_1 is strictly left of B'_1. If $k > k'$, then B_1 is weakly right of B'_1. The result now follows from (a).

(c) Letting T' denote $(T \rightarrow c') \rightarrow c$, Lemma 4.7 implies that C equals the bumping path B of $T' \leftarrow k$ while C' equals the bumping path B' of $(T' \leftarrow k) \leftarrow k'$. Applying (a), it suffices to show that C is strictly left of C' in some row which they both occupy. Consider row r, the top row of C'. Since c is strictly left of c', either C and C' have no rows in common, in which case the result is trivial, or else C has a cell in row r. When we choose the elements of C, we have already performed the reverse row insertion $T \rightarrow c'$. In particular, the cell c' is empty. Therefore, C must be strictly left of C' in row r, as required. The second assertion now follows by applying (b) to these new B and B'. □

To foreshadow the role of Lemma 4.8 in the following section, we give a proof of the classical Pieri rule using this result.

Proof of Theorem 3.1. The formula is proved if we can give a bijection between SSYTs of shape $\lambda * (n)$ and SSYTs of shape λ^+ such that λ^+/λ is a horizontal strip of size n. Let $k_1 \leq \cdots \leq k_n$ be entries of (n) from left to right. Repeated applications of (a) and (b) of Lemma 4.8 ensure that row inserting these entries into an SSYT of shape λ will add a horizontal strip of size n to λ. By

Lemma 4.7, this establishes a bijection where the inverse map is given by reverse row inserting the cells of λ^+/λ from right to left. $\square$

5. A sign-reversing involution

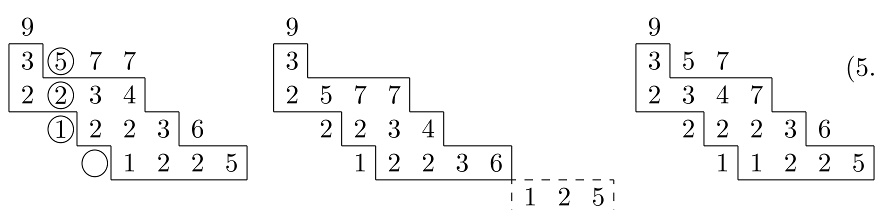
Throughout this section, fix a skew shape λ/μ. We will be interested in SSYTs of shape λ^+/μ^-, where we always assume that λ^+/λ is a horizontal strip, μ/μ^- is a vertical strip, and $|\lambda^+/\lambda| + |\mu/\mu^-| = n$. Our goal is to construct a sign-reversing involution on SSYTs whose shapes take the form λ^+/μ^-, such that the fixed points are in bijection with SSYTs of shape $(\lambda/\mu) * (n)$.

Our involution is reminiscent of the proof of the classical Pieri rule given in Section 4. By Lemma 4.7, reverse row insertion gives a bijective correspondence provided we record the final entry and its landing row. Our strategy, then, is to reverse row insert the cells of λ^+/λ from right to left, recording the entries as we go. If at some stage we land in row $r \geq 1$, we will then re-insert all the previous final entries. More formally, we have the following definition of a *downward slide of* T.

Definition 5.1. Let T be an SSYT of shape λ^+/μ^-. Define the *downward slide of* T, denoted $\mathrm{D}(T)$, as follows: construct $T \to c_1$ where c_1 is the rightmost cell of λ^+/λ, and let k_1 denote the final entry. If k_1 lands in row 0, then continue with c_2 the second rightmost cell of λ^+/λ and k_2 the final entry of $(T \to c_1) \to c_2$. Continue until the first time k_m lands in row $r \geq 1$ and set $m' = m - 1$, or set $m = m' = |\lambda^+/\lambda|$ if no such k_m exists. Then $\mathrm{D}(T)$ is given by

$$(\cdots(((\cdots(T \to c_1) \to c_2 \cdots) \to c_m) \leftarrow k_{m'}) \cdots) \leftarrow k_1.$$

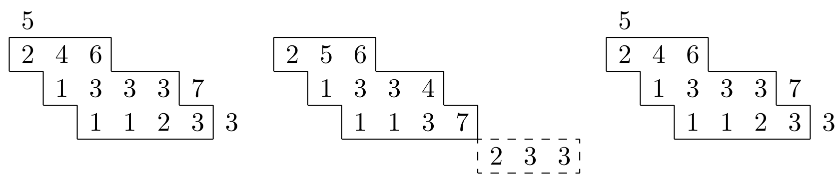
Example 5.2. With T shown on the left below, we exhibit the construction of $\mathrm{D}(T)$ in two steps. We find that $m = 4$ and the middle SSYT shows the result of $(((T \to c_1) \to c_2) \to c_3) \to c_4$. The entries that land in row 0 are recorded in the dashed box. Then the SSYT on the right is $\mathrm{D}(T)$. The significance of the circles will be explained later.



(5.1)

Alternatively, if T is the SSYT shown on the left below, we find that $m = 3$ and that all three final entries land in row 0. Then $m' = |\lambda^+/\lambda|$ and Lemma 4.7 ensures that $D(T) = T$. Below in the middle, we have shown $((T \to c_1) \to c_2) \to c_3$. The position of the dashed box is intended to be suggestive: together with the entries in the outlined shape, we see that we have an SSYT of shape

$(\lambda/\mu) * (n) = (653/21) * (3)$.



The final reverse bumping path in a downward slide will play an important role in the sign-reversing involution. Therefore, with notation as in Definition 5.1, if $m < |\lambda^+/\lambda|$, then we refer to the reverse bumping path of k_m as the *downward path of* T. The cells of the downward path of T are circled above. Say that the downward path of T *exits right* if its bottom cell (which may be empty) is strictly below the bottom cell μ/μ^-. Our terminology is justified since one can show that the exits right condition is equivalent to the bottom cell of the downward path being weakly right of the bottom cell of μ/μ^-. The importance of the exits right condition is revealed by the following result.

Proposition 5.3. *Suppose T is an SSYT of shape λ^+/μ^- such that the downward path of T, if it exists, exits right. Then $\mathrm{D}(T)$ is an SSYT of shape λ'/μ', where λ'/λ (resp. μ/μ') is a horizontal (resp. vertical) strip.*

Proof. First observe that if T has no downward path, then $\mathrm{D}(T) = T$ and clearly has the required shape. Otherwise, using the notation from Definition 5.1, for each $1 \leq i < m$, c_i has a reverse bumping path leading down to the first row. Therefore by Lemma 4.8(c), the reverse bumping path of c_i lies strictly right of that of and c_{i+1}, and $k_{i+1} \leq k_i$ for $1 \leq i < m-1$. After the last reverse row insertion, which is along the downward path, a cell containing k_m is added to the left end of a row strictly below the bottom cell of μ/μ^-. Thus μ/μ' is indeed a vertical strip. By (a) and (b) of Lemma 4.8, since $k_{m-1} \leq \cdots \leq k_1$, reinserting these entries adds a horizontal strip, so it remains to show that the cell added when inserting k_{m-1} lies strictly right of those cells of λ^+/λ that were not moved under the downward slide. This will be the case if the cell added when inserting k_{m-1} lies weakly right of c_m. Indeed, the bumping path of k_{m-1} will either be the reverse bumping path of c_{m-1} or it will have been affected by the changes on the downward path and so will intersect the downward path. In the former case the bumping path will end with the addition of c_{m-1}, while in the latter it ends with the addition of c_m, as required. $\square$

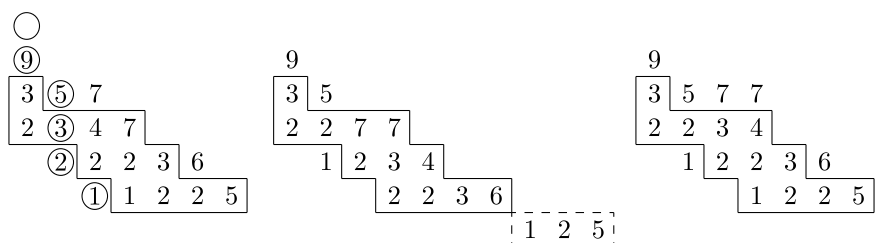
Supposing $\mathrm{D}(S) = T$ with $T \neq S$, the next step is to invert the downward slides for such T. Since any such T necessarily has $\mu^- \neq \mu$, the idea is to internally row insert the bottom cell of μ/μ^-. However, before doing this we must reverse row insert certain cells of λ^+/λ, as in a downward slide. To describe which cells to reverse insert, we define the *upward path of* T to be the bumping path that would result from internal row insertion of the entry in the bottom cell of μ/μ^-. Roughly, we will reverse row insert anything that is weakly right of this upward path.

Definition 5.4. Let T be an SSYT of shape λ^+/μ^- such that $\mu^- \neq \mu$. Define the *upward slide of* T, denoted $\mathrm{U}(T)$, as follows: construct $T \to c_1$ where c_1 is the rightmost cell of λ^+/λ, and let k_1 denote its final entry and B_1 its bumping path. If B_1 fails to stay weakly right of the upward path of T, then set $m = m' = 0$. Otherwise, consider c_2, the second rightmost cell of λ^+/λ, and k_2, the final entry of $(T \to c_1) \to c_2$, and B_2, the corresponding bumping path. Continue until the last time B_m stays weakly right of the upward path of T or until no cell of λ^+/λ remains. Suppose that after the reverse row insertions, the bottom cell of μ/μ^- is in row r and has entry k. Then $\mathrm{U}(T)$ is given by

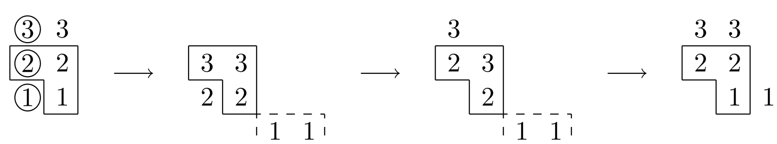
$$(\cdots((((\cdots(T \to c_1) \to c_2 \cdots) \to c_m) \leftarrow_r k) \leftarrow k_{m'}) \cdots) \leftarrow k_1 \tag{5.2}$$

where we set $m' = m$ if k_m lands in row 0, and $m' = m - 1$ otherwise.

Example 5.5. Letting T be the rightmost SSYT of (5.1), the cells of the upward path of T are circled below. We determine $\mathrm{U}(T)$ in three steps. We find that $m = 3$ and the middle SSYT of (5.1) shows $((T \to c_1) \to c_2) \to c_3$. Then $(((T \to c_1) \to c_2) \to c_3) \leftarrow k$ is shown in the middle below, while $\mathrm{U}(T)$ is shown on the right. Comparing with Example 5.2, we observe that the upward slide in this case does indeed invert the downward slide.



There are also instances where the entry k of Definition 5.4 is different from the entry originally at the bottom of the upward path. For example, the same three-step process for constructing $\mathrm{U}(T)$ is shown below for an example with $m = 2$. There, $k = 2$, even though the original upward path of T had 1 as its bottom entry.



As with downward slides and before presenting our involution, we must ensure that the result of an upward slide is always a tableau of the appropriate skew shape.

Proposition 5.6. *Suppose T is an SSYT of shape λ^+/μ^- such that in the upward slide of T, all the final entries of the reverse row insertions land in row 0. Then $\mathrm{U}(T)$ is an SSYT of shape λ'/μ', where λ'/λ (resp. μ/μ') is a horizontal (resp. vertical) strip.*

Proof. If none of the reverse bumping paths $B_1, \dots, B_m$ in the upward slide of T intersect the upward path, then inserting the bottom cell of the upward path into the row above will follow the upward path. So applying U to T will simply amount to row insertion along the upward path, with any movement along the B_i being inverted in the course of applying U. Thus, with notation as in Definition 5.4, we can assume that B_m intersects the upward path of T, where B_m is the last reverse bumping path that stays weakly right of the upward path of T. Note that by Lemma 4.8(c), no other reverse bumping path $B_{m'}$ with $m' < m$ can intersect the upward path. When k is internally inserted into T, it will follow the upward path of T until it intersects B_m from which time it will follow B_m, ultimately adding c_m back to the tableau. Next inserting k_m will result in a bumping path B'_m following B_m until one row below the point of intersection. Note that B'_m cannot intersect the bottom cell of the upward path, since that cell is now empty. Therefore, in the row of the bottom cell of the upward path, B'_m is strictly right of the upward path. Thus we can apply Lemma 4.8(a) to deduce that B'_m will necessarily remain strictly right of the bumping path for k. Finally, by Lemma 4.8(c), we have that $k_m \leq \cdots \leq k_1$. Thus, by (a) and (b) Lemma 4.8, inserting the remaining entries still results in the addition of a horizontal strip. □

Our involution will consist of either applying a downward slide or an upward slide. The decision for which slide to apply is roughly based on which of the downward path of T or the upward path of T lies further to the right.

Definition 5.7. Consider the set of SSYTs T of shape λ^+/μ^- such that that λ^+/λ is a horizontal strip and μ/μ^- is a vertical strip. Define a map ϕ on such T by

$$\phi(T) \;=\; \begin{cases} \mathrm{D}(T) & \text{if } T \text{ has no upward path or} \\ & \text{the downward path of } T \text{ exists and exits right,} \\ \mathrm{U}(T) & \text{otherwise.} \end{cases}$$

Theorem 5.8. *The map ϕ defines an involution on the set of SSYTs with shapes of the form λ^+/μ^- where λ^+/λ is a horizontal strip and μ/μ^- is a vertical strip.*

Proof. We refer the reader to Examples 5.2 and 5.5 for illustrations of the ideas of the proof.

First suppose that $\phi(T) = \mathrm{D}(T)$. By Definition 5.7, the first way in which this can happen is if T has neither an upward path nor a downward path. This corresponds to the case $m' = m$ in Definition 5.1 which, by Lemma 4.7(b), results in $\phi(T) = \mathrm{D}(T) = T$, trivially an involution.

Therefore, assume the downward path of T exists and exits right, so by Proposition 5.3, $\phi(T)$ has the required shape. Moreover, since $\mathrm{D}(T)$ adds a cell to μ/μ^-, $\phi(T)$ must have an upward path. Suppose cell c_m is at the top of the downward path of T. It follows from Lemma 4.8 that all cells strictly right of c_m and above λ in T have reverse bumping paths strictly right of the downward path of T and, by definition of the downward path, their reverse row insertions all land in row 0. For any cell c strictly left of c_m in T, Lemma 4.8(c) implies that the reverse bumping path of c in T must remain strictly left of the downward path of T, and this property persists in $\phi(T)$. Therefore either $\phi(T)$ has no downward path or the downward path does not exit right, and so $\phi(\phi(T)) = \mathrm{U}(\mathrm{D}(T))$.

To show that $\mathrm{U}(\mathrm{D}(T)) = T$, first consider D applied to T. After reverse row inserting along the reverse bumping paths including the downward path, suppose we have arrived at an SSYT S. We next row insert the final entries $k_{m-1}, \ldots, k_1$ of Definition 5.1. These new bumping paths may have been affected by the changes on the downward path but, as shown in the proof of Proposition 5.3, the bumping path B of k_{m-1} still lies weakly right of the downward path. Next, we consider the application of U to $\mathrm{D}(T)$ and observe that, since the downward path of T exits right, it shares a bottom cell with the upward path of $\mathrm{D}(T)$. Thus the upward path must stay weakly left of the downward path and, in particular, B lies weakly right of the upward path. Moreover, B corresponds to the B_m of Definition 5.4 since, as mentioned in the previous paragraph, reverse bumping paths further left will have a bottom cell that is strictly left of the bottom cell of the upward path. A key idea is now evident: after we perform the first part of U by applying the reverse row insertions, we will return to the SSYT S. Therefore, the rest of the application of U to $\mathrm{D}(T)$ will invert the reverse row insertions from the application of D to T, as required.

Next, suppose $\phi(T) = \mathrm{U}(T)$. By Definition 5.7, T has an upward path and either has no downward path or the downward path does not exit right. So, the downward path, if it exists, does not stay weakly right of the upward path. In particular, all the final entries of the reverse row insertions in the upward slide land in row 0. So by Proposition 5.6, $\phi(T)$ has the required shape.

If none of the reverse bumping paths $B_1, \ldots, B_m$ in the upward slide of T intersect the upward path then, as we observed in the proof of Proposition 5.6, applying U to T will simply amount to row insertion along the upward path, with any movement along the B_i being inverted in the course of applying U. In particular, the upward path of T will become the downward path of $\mathrm{U}(T)$ and hence it exits right. Again, the downward path of $\mathrm{U}(T)$ will not intersect the other reverse bumping paths of $\mathrm{U}(T)$, which will still be $B_1, \ldots, B_m$. So applying D to $\mathrm{U}(T)$ will simply amount to the reverse row insertion along the downward path. Thus $\phi(\phi(T)) = \mathrm{D}(\mathrm{U}(T)) = T$.

On the other hand, if B_m from Definition 5.4 intersects the upward path, we know from the proof of Proposition 5.6 that internally inserting the bottom entry k of the upward path into the row above will eventually follow B_m and all other insertions will have bumping paths $B'_m, B'_{m-1}, \ldots, B'_1$ strictly to the right. Therefore this bumping path for k will become the downward path of $\mathrm{U}(T)$ and again it clearly exits right. Thus when we apply D to $\mathrm{U}(T)$, we will first reverse row insert along the B'_i and then invert the changes caused by the internal insertion of k. As a result, when we complete the D operation by reinserting the final entries of $B'_m, B'_{m-1}, \ldots, B'_1$, the bumping paths will be the same as the original reverse bumping paths when we applied U to T. Thus $\phi(\phi(T)) = \mathrm{D}(\mathrm{U}(T)) = T$. $\square$

We now have all the ingredients needed to prove the skew Pieri rule.

Proof of Theorem 3.2. Using the expansion of s_{λ^+/μ^-} in terms of SSYTs as in (2.1), observe that if $\phi(T) \neq T$, then the T and $\phi(T)$ occur with different signs in the right-hand side of (3.3). Since ϕ clearly preserves the monomial associated to an SSYT, the monomials corresponding to T and $\phi(T)$ in the right-hand side of (3.3) will cancel out. Because $s_{\lambda/\mu}s_n = s_{(\lambda/\mu)*(n)}$, it remains to show that there is a monomial-preserving bijection from fixed points of ϕ to SSYTs of shape $(\lambda/\mu) * (n)$.

Note that T is a fixed point of ϕ only if T has neither an upward path nor a downward path. This happens if and only if $\mu^- = \mu$ and when reverse row inserting the cells of λ^+/λ from right to left, every final entry lands in row 0. In particular, the entries of T remaining after reverse row inserting the cells of λ^+/λ form an SSYT of shape λ/μ. Say the final entries of the reverse row insertions are $k_n, \ldots, k_1$ in the order removed. By Lemma 4.8(c), since λ^+/λ is a horizontal strip, we have $k_1 \leq \cdots \leq k_n$ and so these entries form an SSYT of shape (n). By Lemma 4.7, this process is invertible and therefore establishes the desired bijection. □

6. Concluding remarks

6.1. Littlewood–Richardson fillings

We proved Theorem 3.2 by working with SSYTs. In particular, we showed that the two sides of (3.3) were equal when expanded in terms of monomials. Alternatively, we could consider the expansions of both sides of (3.3) in terms of Schur functions. The Littlewood–Richardson rule states that the coefficient of s_ν in the expansion of any skew Schur function $s_{\lambda/\mu}$ is the number of *Littlewood–Richardson fillings* (*LR-fillings*) of shape λ/μ and content ν. (The interested reader unfamiliar with LR-fillings can find the definition in [22].) It is not hard to check that our maps D and U send LR-fillings to LR-fillings, and bumping within LR-fillings has some nice properties. For example, the entries along a (reverse) bumping path are always $1, 2, \ldots, r$ from bottom to top for some r.

However, we chose to give our proof in terms of SSYTs because one does not need to invoke the power of the Littlewood–Richardson rule to prove the classical Pieri rule, and we wanted the same to apply to the skew Pieri rule.

6.2. The bigger picture

We would like to conclude by asking if the skew Pieri rule can be shown to be a special case of a larger framework. For example, [2] and [6] both give general setups that might be relevant. It would be of obvious interest if these frameworks or any others in the literature could be used to rederive the skew Pieri rule.

In a similar spirit, we close with a conjectural rule for the product $s_{\lambda/\mu}s_{\sigma/\tau}$, which has now been proved by Lam, Lauve and Sottile in [7]. This rule gives the skew Pieri rule when $\sigma/\tau = (n)$ and gives the classical Littlewood–Richardson rule when $\mu = \tau = \emptyset$. The Littlewood–Richardson rule can itself be used to evaluate the general product $s_{\lambda/\mu} s_{\sigma/\tau}$, but our rule will be different in that, like the skew Pieri rule, we will be adding boxes to both the inside and outside of λ/μ.

As before, fix a skew shape λ/μ, and suppose we have a skew shape λ^+/μ^- such that $\lambda^+ \supseteq \lambda$ and $\mu^- \subseteq \mu$. We no longer require that λ^+/λ (resp. μ/μ^-) is a horizontal (resp. vertical) strip. We will need a few new definitions that are variants of those that arise in the Littlewood–Richardson rule. Let T^+ be an SSYT of shape λ^+/λ and let T^- be a filling of μ/μ^-. We let T^- be an *anti-semistandard Young tableau* (ASSYT), meaning that the entries of T^- strictly decrease from left-to-right along rows, and weakly decrease up columns. When $\lambda/\mu = 7541/33$ and $\lambda^+/\mu^- = 9953/1$, an example of a pair (T^-, T^+) with the above-stated properties is

$$\begin{array}{ccccccccc} & 5 & 6 & & & & & & \\ & & & & 3 & & & & \\ 5 & 3 & 1 & & & 1 & 4 & 4 & 5 \\ & 3 & 2 & & & & & 2 & 4 \end{array} \quad . \tag{6.1}$$

The reverse reading word of the pair (T^-, T^+) is the sequence of entries obtained by first reading the entries of T^- from bottom-to-top along its columns, starting with rightmost column and moving left, followed by reading the entries of T^+ from right-to-left along its rows, starting with the bottom row and moving upwards. The reverse reading word of the example above is 21335425441365. A word w is said to be Yamanouchi if, in the first j letters of w, the number of occurrences of i is no less than the number of occurrences of $i+1$, for all i and j. For a partition τ, a word w is said to be τ-Yamanouchi if it is Yamanouchi when prefixed by the following concatenation: τ_1 1's, followed by τ_2 2's, and so on. The reverse reading word of (6.1) is certainly not Yamanouchi but is 5321-Yamanouchi.

For a proof of the following conjecture, see Theorem 6 and Remark 7 in [7].

Conjecture 6.1. *For any skew shapes λ/μ and σ/τ,*

$$s_{\lambda/\mu} s_{\sigma/\tau} = \sum_{\substack{T^- \in \mathrm{ASSYT}(\mu/\mu^-) \\ T^+ \in \mathrm{SSYT}(\lambda^+/\lambda)}} (-1)^{|\mu/\mu^-|} s_{\lambda^+/\mu^-},$$

where the sum is over all ASSYTs T^- of shape μ/μ^- for some $\mu^- \subseteq \mu$, and SSYTs T^+ of shape λ^+/λ for some $\lambda^+ \supseteq \lambda$, with the following properties:

- *the combined content of T^- and T^+ is the component-wise difference $\sigma - \tau$, and*
- *the reverse reading word of (T^-, T^+) is τ-Yamanouchi.*

For example, to the product $s_{7541/33}\, s_{755431/5321}$, the pair (T^-, T^+) of (6.1) contributes $-s_{9953/1}$. Note that when $\mu = \tau = \emptyset$, Conjecture 6.1 is exactly the classical Littlewood–Richardson rule. If instead $\sigma/\tau = (n)$, then all the entries of T^- and T^+ must be 1's, and the skew Pieri rule results.

Another interesting special case of Conjecture 6.1 is when σ/τ is a horizontal strip with row lengths $\rho = \sigma - \tau$ from bottom to top, with ρ a partition. In this case, $s_{\sigma/\tau} = h_\rho$, while the τ-Yamanouchi property is trivially satisfied for any (T^-, T^+) with content $\sigma - \tau$. Conjecture 6.1 then gives the following expression for the product $s_{\lambda/\mu} h_\rho$ for any skew shape λ/μ and partition ρ:

$$s_{\lambda/\mu} h_\rho = \sum_{\substack{T^- \in \mathrm{ASSYT}(\mu/\mu^-) \\ T^+ \in \mathrm{SSYT}(\lambda^+/\lambda)}} (-1)^{|\mu/\mu^-|} s_{\lambda^+/\mu^-},$$

where the sum is over all ASSYTs T^- of shape μ/μ^- for some $\mu^- \subseteq \mu$, and SSYTs T^+ of shape λ^+/λ for some $\lambda^+ \supseteq \lambda$, such that the combined content of T^- and T^+ is ρ. Observe that this result follows directly from repeated applications of the skew Pieri rule.

7. Acknowledgments

We are grateful to a number of experts for informing us that they too were surprised by the existence of the skew Pieri rule, and particularly to Richard Stanley for providing an algebraic proof of the $n = 1$ case that preceded our combinatorial proof. We are also grateful to Thomas Lam for extending Stanley's proof to the general case and for his willingness to append his proof to this manuscript. This research was performed while the second author was visiting MIT, and he thanks the mathematics department for their hospitality. Conjecture verification was performed using [1, 23].

A. An algebraic proof (by Thomas Lam)

Let $\langle ., . \rangle : \Lambda \times \Lambda \to \mathbb{Q}$ denote the (symmetric, bilinear) Hall inner product. For $f \in \Lambda$, we let $f^\perp : \Lambda \to \Lambda$ denote the adjoint operator to multiplication by f, so that $\langle fg, h \rangle = \langle g, f^\perp h \rangle$ for $g, h \in \Lambda$.

Proposition A.1. *For $n \geq 1$, and $f, g \in \Lambda$, we have*

$$f\, h_n^\perp(g) = \sum_{k=0}^{n} (-1)^k h_{n-k}^\perp (e_k^\perp(f) g). \tag{A.1}$$

Proof. We shall use the formula [12, 2.6′], for $n \geq 1$

$$\sum_{i=0}^{n} (-1)^i e_i h_{n-i} = 0 \tag{A.2}$$

and [12, Example I.5.25(d)]

$$h_n^\perp(fg) = \sum_{i=0}^{n} h_{n-i}^\perp(f) h_i^\perp(g). \tag{A.3}$$

In the following we shall use the fact that the map $\Lambda \to \mathrm{End}_{\mathbb{Q}}(\Lambda)$ given by $f \to f^\perp$ is a ring homomorphism [12, Example I.5.3]. Starting from the right-hand side of (A.1) and using (A.3) we have

$$\sum_{k=0}^{n}(-1)^k h_{n-k}^\perp(e_k^\perp(f)g) = \sum_{k=0}^{n}(-1)^k \sum_{j=0}^{n-k} h_j^\perp(e_k^\perp(f))h_{n-k-j}^\perp(g).$$

Under the substitution $n-k=j+i$, the right-hand side becomes

$$\sum_{i=0}^{n}(-1)^{n-i}\left(\sum_{j=0}^{n-i}(-1)^j(h_j^\perp e_{n-j-i}^\perp)(f)\right)h_i^\perp(g). \tag{A.4}$$

Since $\sum_{j=0}^{n-i}(-1)^j(h_j^\perp e_{n-j-i}^\perp)(f)=0$ for $n-i>0$ using $(\mathrm{A.2})^\perp$, but is equal to f for $n=i$, we see that (A.4) reduces to $f\,h_n^\perp(g)$. □

Proof of Theorem 3.2. It is well known (see [12, I.(4.8), I.(5.1)] or [22, Corollary 7.12.2, (7.60)]) that for two partitions λ and μ, we have $\langle s_\lambda, s_\mu\rangle = \delta_{\lambda,\mu}$ and $s_\mu^\perp s_\lambda = s_{\lambda/\mu}$, where $s_{\lambda/\mu}=0$ if $\mu \not\subseteq \lambda$. Let $g\in\Lambda$. We calculate

$$\begin{aligned}\langle s_{\lambda/\mu}h_n, g\rangle = \langle s_{\lambda/\mu}, h_n^\perp g\rangle = \langle s_\mu^\perp s_\lambda, h_n^\perp g\rangle &= \langle s_\lambda, s_\mu h_n^\perp g\rangle \\ &= \langle s_\lambda, \sum_{k=0}^{n}(-1)^k h_{n-k}^\perp(e_k^\perp(s_\mu)g)\rangle\end{aligned}$$

by Proposition A.1. Using the Pieri rule (Theorem 3.1), this amounts to

$$\left\langle \sum_{k=0}^{n}(-1)^k \sum_{\substack{\lambda^+/\lambda\ (n-k)\text{-hor. strip}\\ \mu/\mu^-\ k\text{-vert. strip}}} s_{\lambda^+/\mu^-}\ , g\right\rangle.$$

Since the Hall inner product is non-degenerate, we obtain (3.3). □

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