Principal Minor Sums of $(A+tB)^m$

Charles R. Johnson *
Stefan Leichenauer †
Peter McNamara ‡
Roberto Costas §

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Abstract

The question is raised whether the sum of the k-by-k principal minors of the titled matrix is a polynomial (in t) with positive coefficients, when A and B are positive definite. This would generalize a conjecture made by Bessis-Moussa-Villani, as stated by Lieb and Seiringer [LS]. We give a variety of evidence for this further question, some of which is new.

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*Department of Mathematics, College of William and Mary, Williamsburg, VA 23187, USA. (crjohnso@math.wm.edu)

†Department of Mathematics, Brown University, Providence, RI 02912, USA. (swl@brown.edu)

‡Laboratoire de Combinatoire et d'Informatique Mathématique, Université du Québec à Montréal, Montréal (Québec) H3C 3P8, Canada. (mcnamara@lacim.uqam.ca)

§Departmento de Mathemáticas, Universidad Carlos III de Madrid, 28911 Leganés (Madrid), Spain. (rcostas@math.uc3m.es)

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For an n-by-n matrix X over a field, let $S_{k,m,n}(X)$ denote the sum of the k-by-k principal minors of X^m. Our primary interest is in $S_{k,m,n}(A+tB)$ when A and B are positive definite. In this event, $S_{k,m,n}(A+tB)$ is a polynomial, $s_{k,m}(t)$, of degree km in t. We conjecture that this polynomial has all positive coefficients (the "positivity conjecture"). We are motivated, in part, by the fact that the special case $S_{1,m,n}$ of this conjecture is equivalent to the still open conjecture of [BMV] as noted in [HiJ2] and proven in [LS]. (Several special cases of the BMV conjecture have been proven by Hillar and Johnson - see references.) Our purpose is to present the existing evidence for this conjecture, including several new special cases proven here. In general, we feel that focus upon this conjecture is valid for a variety of reasons, including that a number of determinantal inequalities are implied. It should be noted that it can easily happen that not all constituent summands of $S_{k,m,n}(A+tB)$ are polynomials with positive coefficients.

We summarize the cases (by k,m,n) of the positivity conjecture that have been verified, including those first demonstrated herein:

(1) $k = n$ (any m);

(2) $n < 3$ (any $k \leq n$, any m)

(3) $m < 3$ (any k,n);

(4) $k = 1$, $m < 6$ (any n); and

(5) $k = 1$, $m = 6$, $n = 3$.

Case (4) has been verified in [HiJ1]. Case (5) has recently been verified in [HiJ2] and involves an intricate calculation, quite different from case (4). Another verification may be given using (in part) M-matrices. Regarding (2), the subcase $k = 1$, $n = 2$ has been verified in [HiJ1]; since the case $n = 1$ is trivial, this leaves the subcase $k = 2$, $n = 2$, which is a subcase of (1). This leaves cases (1) and (3) to be verified, which we do here. First, we consider case(1).

Theorem 1. *If A and B are n-by-n positive definite matrices and m a positive integer, then all coefficients of the the degree nm polynomial*

$$s_{n,m}(t) = \det[(A+tB)^m]$$

are positive.

Proof. Since the function det is multiplicative, it suffices to prove the claim in case $m = 1$; in the general case, $s_{n,m}(t)$ is then a product of polynomials with positive coefficients. Suppose $m = 1$. The positive definite matrices

A and B may be simultaneously diagonalized by congruence (as in [HoJ, Theorem 7.6.4]), so that for some nonsingular C,

$$A + tB = C^*(D_A + tD_B)C$$

in which D_A and D_B are diagonal matrices with positive diagonal entries. Then,

$$\det(A + tB) = \det C^*(D_A + tD_B)C = \det CC^* \det(D_A + tD_B).$$

Since $\det CC^* > 0$, it suffices to show that $\det(D_A + tD_B)$ has positive coefficients. But, if $D_A = \operatorname{diag}(a_1, \ldots, a_n)$ and $D_B = \operatorname{diag}(b_1, \ldots, b_n)$, $\det(D_A + tD_B)$ is the product

$$\prod_{i=1}^{n}(a_i + tb_i),$$

of linear polynomials, each of which has positive coefficients, completing the proof. □

It remains to consider case (3).

Theorem 2. *If A and B are n-by-n positive definite matrices, $m < 3$ is a positive integer, and $0 < k \leq n$ is a positive integer, then the degree km polynomial*

$$s_{k,m}(t) = \sum_{|\alpha|=k} \det\{(A + tB)^m[\alpha]\}$$

has all positive coefficients.

Proof. First consider the case $m = 1$. In this case we have

$$s_{k,1}(t) = \sum_{|\alpha|=k} \det\{(A + tB)[\alpha]\}.$$

Now, since $A[\alpha]$ and $B[\alpha]$ are principal submatrices of positive definite matrices, they are positive definite themselves. But then, since $(A + tB)[\alpha] = A[\alpha] + tB[\alpha]$, Theorem 1 tells us that each $\det\{(A + tB)[\alpha]\}$ is a polynomial in t with positive coefficients. Therefore their sum is also a polynomial in t with positive coefficients.

Now consider the case $m = 2$:

$$s_{k,2}(t) = \sum_{|\alpha|=k} \det\{(A + tB)^2[\alpha]\}.$$

As before, there exists a nonsingular C such that $A+tB = C(D_A+tD_B)C^*$ where D_A and D_B are diagonal with positive entries. Therefore we have

$$s_{k,2}(t) = \sum_{|\alpha|=k} \det\{(C(D_A+tD_B)C^*)^2[\alpha]\}$$

and, by performing a similarity by C^*,

$$s_{k,2}(t) = \sum_{|\alpha|=k} \det\{(C^*C(D_A+tD_B))^2[\alpha]\}.$$

Define the positive definite matrix L by $L = C^*C$ and the t-dependent diagonal matrix P by $P = D_A+tD_B$. With this notation the above is just

$$s_{k,2}(t) = \sum_{|\alpha|=k} \det\{(LP)^2[\alpha]\}.$$

We use the Cauchy-Binet formula to compute the minors of $(LP)^2$ and find

$$\det\{(LP)^2[\alpha]\} = \sum_{|\beta|=k} \det\{(LP)[\alpha,\beta]\} \det\{(LP)[\beta,\alpha]\}.$$

We invoke Cauchy-Binet once more to get

$$\det\{(LP)^2[\alpha]\} = \sum_{|\beta|=k} \left[\sum_{|\gamma|=k} \det\{L[\alpha,\gamma]\} \det\{P[\gamma,\beta]\}\right] \left[\sum_{|\mu|=k} \det\{L[\beta,\mu]\} \det\{P[\mu,\alpha]\}\right].$$

Since P is diagonal, we know that only the principal minors of P are nonzero. Therefore $\gamma = \beta$ and $\mu = \alpha$. This simplifies the above to

$$\det\{(LP)^2[\alpha]\} = \sum_{|\beta|=k} \det\{P[\beta]\} \det\{L[\alpha,\beta]\} \det\{L[\beta,\alpha]\} \det\{P[\alpha]\}.$$

Since L is positive definite, it is in particular Hermitian. Therefore $\det\{L[\beta,\alpha]\} = \det\{L^*[\beta,\alpha]\} = \overline{\det\{L[\alpha,\beta]\}}$. Thus we have

$$\det\{(LP)^2[\alpha]\} = \sum_{|\beta|=k} \det\{P[\beta]\} |\det\{L[\alpha,\beta]\}|^2 \det\{P[\alpha]\},$$

and since all summands are now positive, the polynomial $\det\{(LP)^2[\alpha]\}$ will have positive coefficients and the result follows. $\square$

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