P-partitions and Quasi-Symmetric Functions

York University Applied Algebra Seminar
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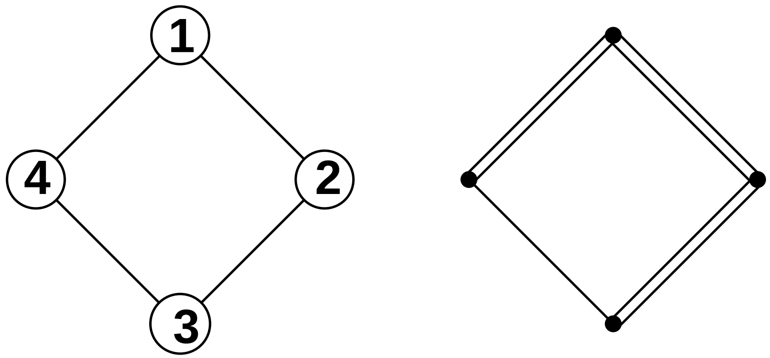
Outline

- Introduction and Stanley's Conjecture
- Malvenuto's reformulation
- Cylindric skew shapes
- Conjecture true in "most" cases
- Open problems

P-partitions

P: partially ordered set (poset)
$\omega : P \to \{1, 2, \ldots, |P|\}$ bijective labelling



DEFINITION (R. Stanley) Given a labelled poset (P, ω), a (P, ω)-partition is a map $f : P \to \mathbb{P}$ with the following properties:

- f is *order-preserving*: If $x \leq y$ in P then $f(x) \leq f(y)$
- If $x \lessdot y$ in P and $\omega(x) > \omega(y)$ then $f(x) < f(y)$

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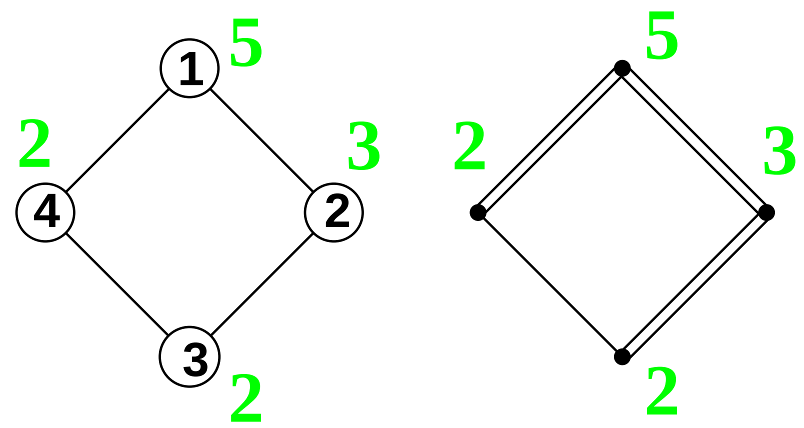

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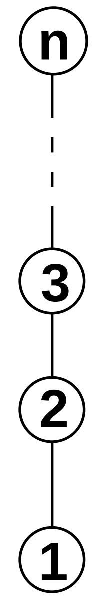
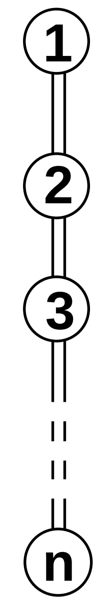
3 motivating examples





- (P, ω) is an antichain: (P, ω)-partition = composition
- (P, ω) is a chain of weak edges: (P, ω)-partition = partition
- (P, ω) is a chain of strict edges: (P, ω)-partition = partition with distinct parts

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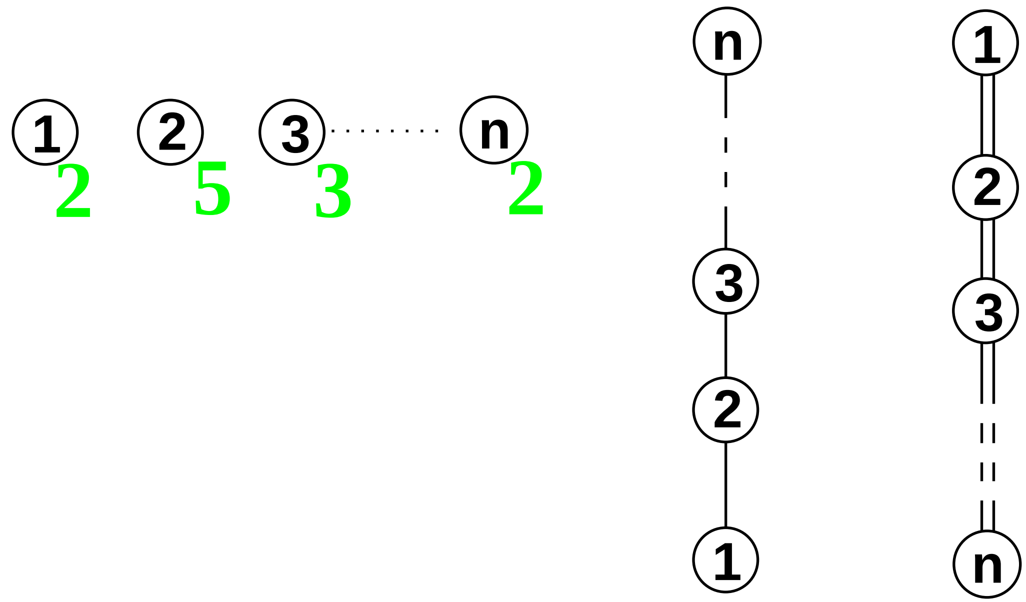

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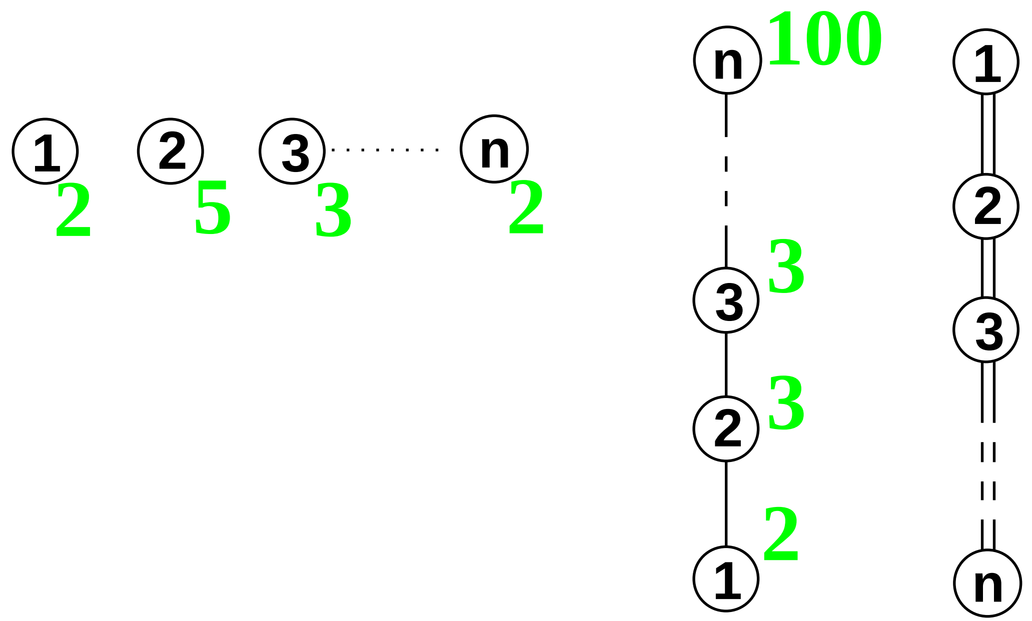

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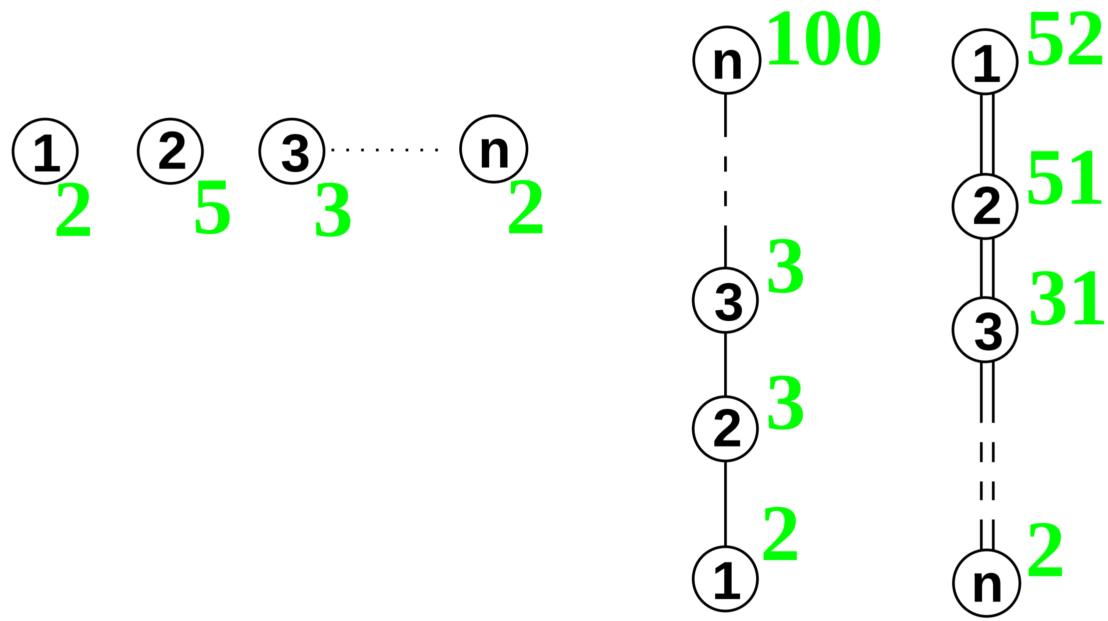

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… *and Quasi-Symmetric Functions*

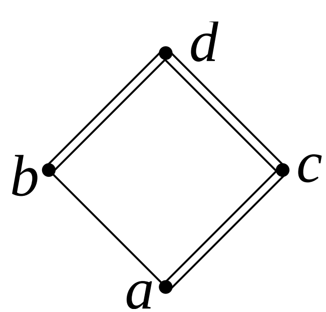
NOTE $K_{P,\omega}(x)$ is a quasi-symmetric function:

Coeff. of $x_{i_1}^{m_1} x_{i_2}^{m_2} \cdots x_{i_k}^{m_k}$ = Coeff. of $x_{j_1}^{m_1} x_{j_2}^{m_2} \cdots x_{j_k}^{m_k}$

whenever $i_1 < i_2 < \cdots < i_k$ and $j_1 < j_2 < \cdots < j_k$.

$$M_{(\alpha_1,\ldots,\alpha_k)} = \sum_{i_1 < \cdots < i_k} x_{i_1}^{\alpha_1} \cdots x_{i_k}^{\alpha_k}$$

EXAMPLE



$f(a) = f(b) < f(c) < f(d)$	M_{211}
$f(a) < f(b) = f(c) < f(d)$	M_{121}
$f(a) < f(b) < f(c) < f(d)$	M_{1111}
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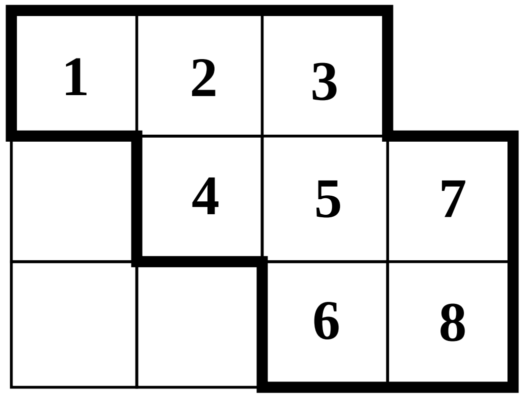
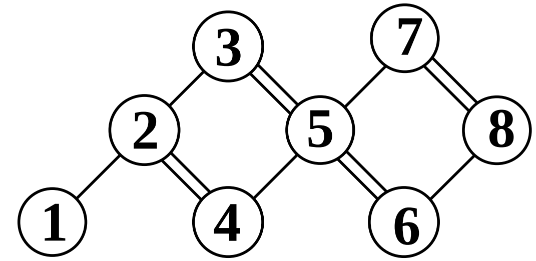
Schur labelled skew shape posets

EXAMPLE





Bijection: SSYT of shape $\lambda/\mu \leftrightarrow (P,\omega)$-partitions
Furthermore,

$$K_{P,\omega}(x) = s_{\lambda/\mu}.$$

BIG QUESTION *What other labelled posets (P,ω) have symmetric $K_{P,\omega}(x)$?*

Stanley's P-partitions Conjecture

CONJECTURE *(Stanley, c.1971) $K_{P,\omega}(x)$ is symmetric if and only if (P,ω) is isomorphic to a Schur labelled skew shape poset.*

- True if all edges are weak: [3] exercise in EC1.
- Stembridge 1993/4: true if $|P| \leq 7$.

SPECIAL CASE Polyominoes

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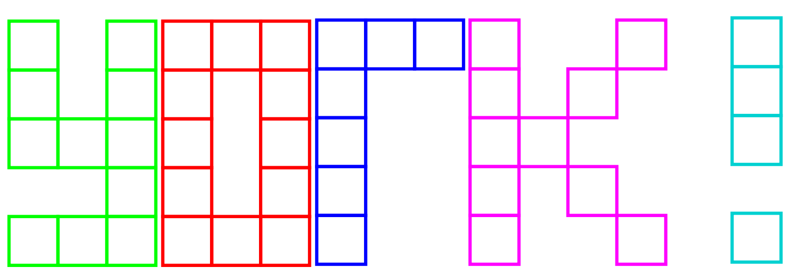
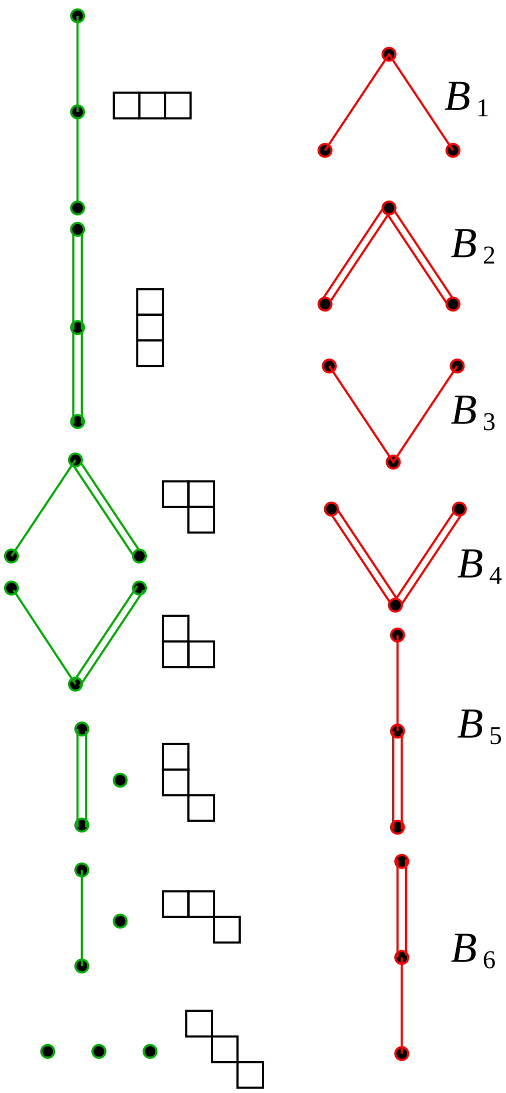


Local Structure of Skew Shape posets



(P,ω)	$K_{P,\omega}(x)$
B_1	$M_3 + M_{21} + 2M_{12} + 2M_{111}$
B_2	$M_{21} + 2M_{111}$
B_3	$M_3 + 2M_{21} + M_{12} + 2M_{111}$
B_4	$M_{12} + 2M_{111}$
B_5	$M_{12} + M_{111}$
B_6	$M_{21} + M_{111}$

NOTE All 3 element convex subposets of a (Schur labelled) skew shape poset must be of one of the green forms, i.e. a skew shape poset cannot have any forbidden convex subposets.

Malvenuto's reformulation

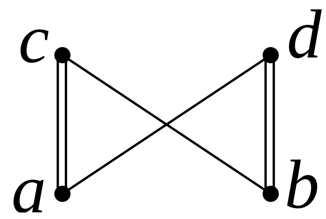
THEOREM *(C. Malvenuto, c. 1995) If a labelled poset (P, ω) has no forbidden convex subposets, then (P, ω) is isomorphic to a skew shape poset.*

In other words, being a skew shape poset is equivalent to having no forbidden convex subposets.

CONJECTURE *(Stanley's conjecture restated) If $K_{P,\omega}$ is symmetric, then (P, ω) has no forbidden convex subposets.*

Arbitrary strict and weak edges

EXAMPLE



$\omega(a) > \omega(c) > \omega(b) > \omega(d) > \omega(a)$ Yikes! Oriented Poset

$K_{P,O}(x)$ = $M_{22} + 2M_{211} + 2M_{121} + 2M_{112} + 4M_{1111} \Rightarrow$ Symmetric. So does it obey Stanley's conjecture?

Arbitrary strict and weak edges

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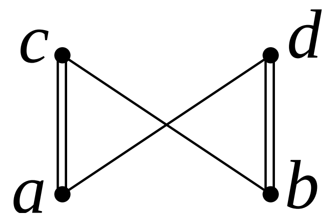

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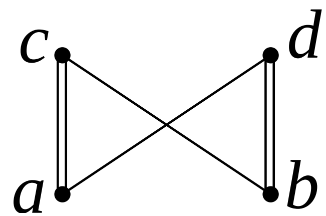

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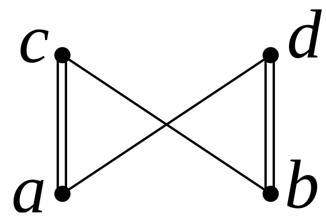

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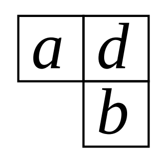

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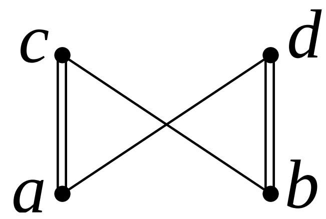

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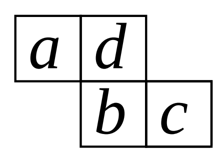

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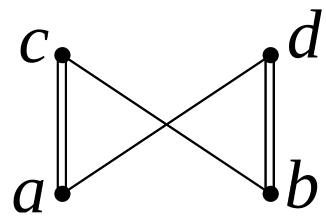

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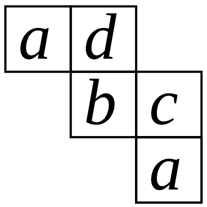

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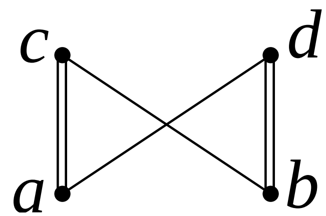

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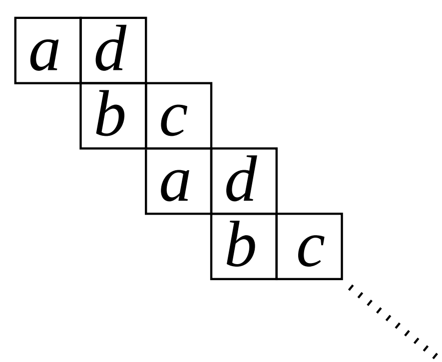

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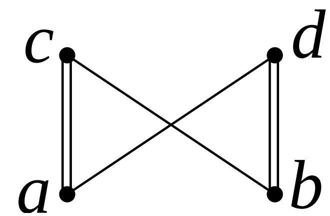

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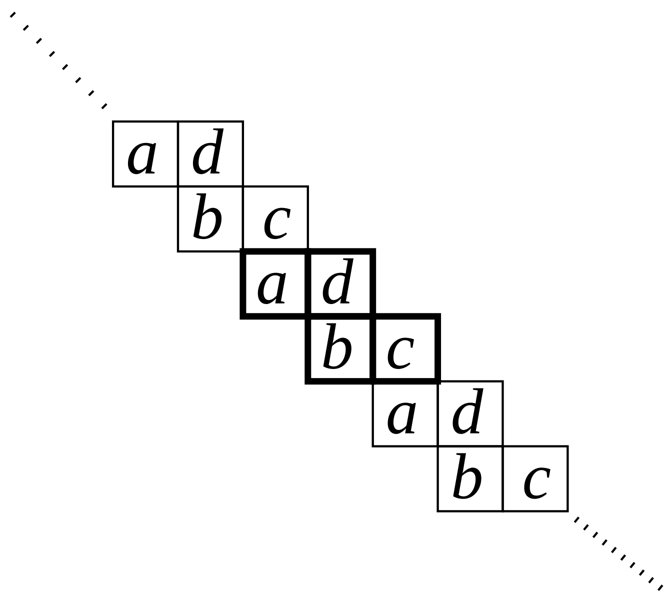


Cylindric skew shapes: definition

Follow Postnikov.
Cylindric partitions: Gessel and Krattenthaler
Proper tableaux: Bertram, Ciocan-Fontanine, Fulton

Fix $u, v \geq 2$.

$$\mathfrak{C}_{uv} = \mathbb{Z}^2/(-u, v)\mathbb{Z}.$$

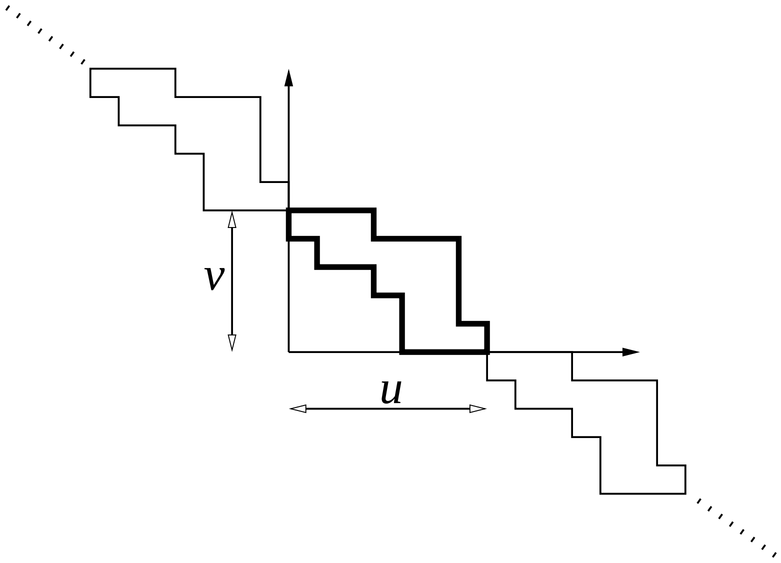
Let $\langle i, j\rangle = (i, j) + (-u, v)\mathbb{Z}$.
We get a partial order on $\mathfrak{C}_{uv}$ from the covering relations:

weak: $\langle i, j\rangle \lessdot \langle i+1, j\rangle$ and strict: $\langle i, j\rangle \lessdot \langle i, j+1\rangle$.

DEFINITION A cylindric skew shape is a finite convex subposet of the poset $\mathfrak{C}_{uv}$.

Cylindric skew shapes: examples

EXAMPLE



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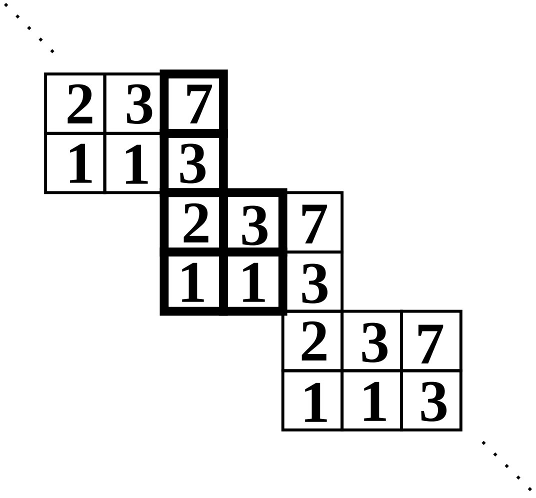

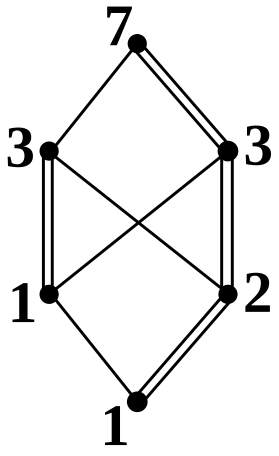

Cylindric skew shapes: results

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CONJECTURE *Let (P, O) be any oriented poset. Then $K_{P,O}(x)$ is symmetric if and only if every connected component of (P, O) is isomorphic to a cylindric skew shape poset.*

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THEOREM *Let (P, O) be an oriented poset. Every connected component of (P, O) is isomorphic to a cylindric skew shape poset if and only if (P, O) has no forbidden convex subposets.*

Open Problems

- Stanley's Conjecture, and its extension.
- Show $K_{P,\omega}(x)$ symmetric $\Rightarrow K_{P,\omega}(x)$ Schur-positive
- ? Is the map $(P,\omega) \to K_{P,\omega}(x)$ injective (modulo rotation of skew shapes) ?
- What about $(P,O) \to K_{P,O}(x)$?
- Given a quasi-symmetric function $K(x)$, how do you tell if $K = K_{P,\omega}(x)$ for some (P,ω) ?

Fast construction of $K_{P,\omega}(x)$

Define a new basis F_α by:

$$F_\alpha = \sum_\beta M_\beta$$

where the sum is over all compositions β than are *finer* than α. E.g., $F_{13} = M_{13} + M_{112} + M_{121} + M_{1111}$.
Then

$$K_{P,\omega}(x) = \sum_{\pi \in \mathcal{L}(P,\omega)} F_{\mathrm{comp}(\pi)}(x).$$

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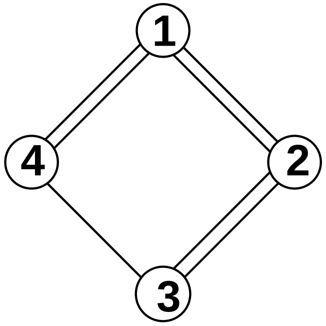

3 4 2 1 3 2 4 1

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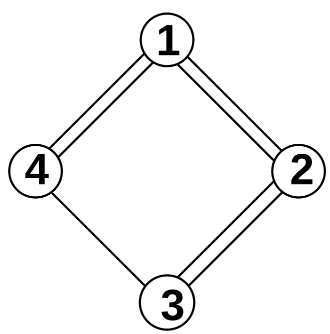

3 4 | 2 | 1   **3 | 2 4 | 1**

2 1 1   **1 2 1**

$$K_{P,\omega}(x) = F_{211} + F_{121} = M_{211} + M_{121} + 2M_{1111}.$$

More open problems

- Given a quasi-symmetric function $K(x)$, how do you tell if $K = K_{P,O}(x)$ for some (P, O) ?
- When is $K_{P,O}(x)$ F-positive?

CONJECTURE $K_{P,O}(x)$ *is* F*-positive if and only if* $(P, O) = (P, \omega)$ *for some* ω*.*

PROPOSITION $K_{P,O}(x)$ *is not* F*-positive if* (P, O) *consists of exactly one cycle.*

- When (P, O) is a cylindric skew shape, show that $K_{P,O}(x)$ is Schur-positive if and only if (P, O) is a skew shape.