

The Möbius function of generalized subword order

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Bucknell University

Joint work with:
Bruce Sagan
Michigan State University

University of Minnesota Combinatorics Seminar

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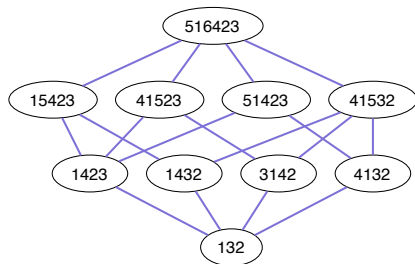
Slides, handout and paper (*Adv. Math.*) available from
www.facstaff.bucknell.edu/pm040/

- ▶ Generalized subword order and related posets
- ▶ Main result
- ▶ Applications
- ▶ Mini-tutorial on discrete Morse theory for posets
- ▶ How DMT gives the proof of main result

Motivation: Wilf's question

Pattern order: order permutations by pattern containment.

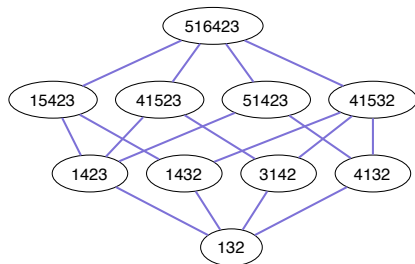
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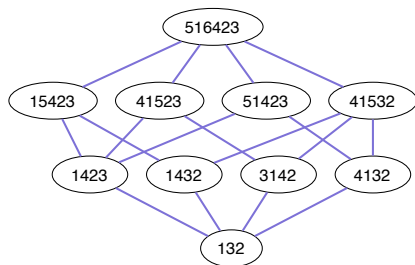


Wilf (2002): What can be said about the Möbius function $\mu(\sigma, \tau)$ of the pattern poset?

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Wilf (2002): What can be said about the Möbius function $\mu(\sigma, \tau)$ of the pattern poset?

- ▶ Sagan & Vatter (2006)
- ▶ Steingrímsson & Tenner (2010)
- ▶ Burstein, Jelínek, Jelínková & Steingrímsson (2011)

Still open.

Motivation for generalized subword order

Our focus: a different poset's Möbius function;
tangentially related to Wilf's question.

Motivation for generalized subword order

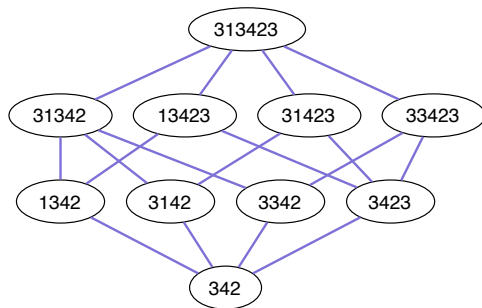
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2 partial orders.

1. Subword order.

A^* : set of finite words over alphabet A .

$u \leq w$ if u is a subword of w , e.g., $342 \leq 313423$.

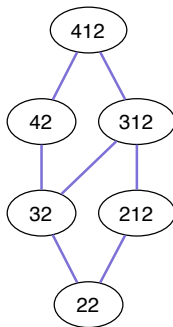


Motivation for generalized subword order

2. An order on compositions.

$(a_1, a_2, \dots, a_r) \leq (b_1, b_2, \dots, b_s)$ if there exists a subsequence $(b_{i_1}, b_{i_2}, \dots, b_{i_r})$ such that $a_j \leq b_{i_j}$ for $1 \leq j \leq r$.

e.g. $2\bar{2} \leq 4\bar{1}\bar{2}$.

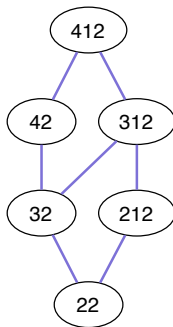


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e.g. $22 \leq 412$.



Composition order \cong pattern order on *layered* permutations

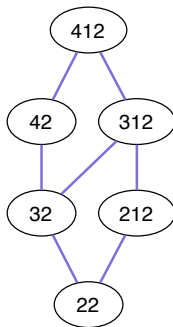
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Main Definition. $u \leq w$ if there exists a subword $w(i_1)w(i_2) \cdots w(i_r)$ of w of the same length as u such that

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Example 1. If P is an antichain, $u(j) \leq_P w(i_j)$ iff $u(j) = w(i_j)$.



Gives subword order on the alphabet P , e.g., $342 \leq 313423$.

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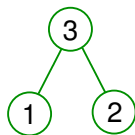


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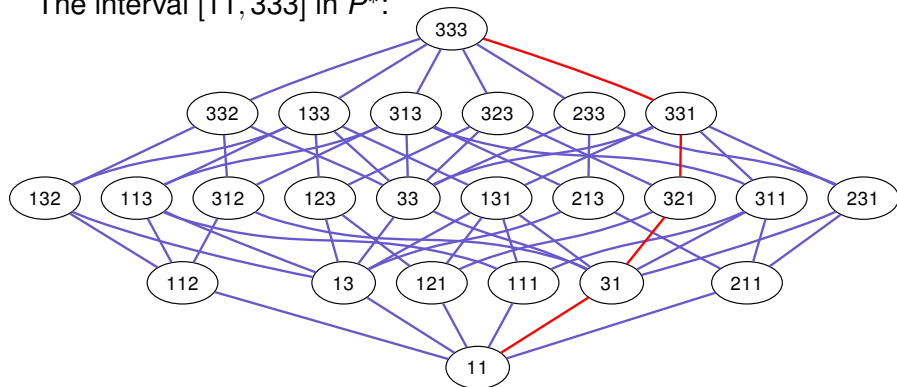
Definition from Sagan & Vatter (2006); appeared earlier in context of well quasi-orderings [Kruskal, 1972 survey].

Key example

Example 3. $P = \Lambda$



The interval $[11, 333]$ in P^* :



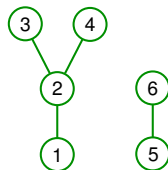
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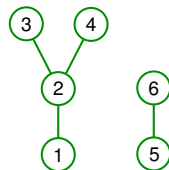


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- ▶ Sagan & Vatter (2006): when $P = \Lambda$, conjecture that $\mu(1^i, 3^j)$ equals certain coefficients of Chebyshev polynomials of the first kind.

Tomie (2010): proof using ad-hoc methods.

Our first goal: a more systematic proof.

Main result

P_0 : P with a bottom element 0 adjoined.

μ_0 : Möbius function of P_0 .

Theorem. Let P be a poset so that P_0 is locally finite. Let u and w be elements of P^* with $u \leq w$. Then

$$\mu(u, w) = \sum_{\eta} \prod_{1 \leq j \leq |w|} \begin{cases} \mu_0(\eta(j), w(j)) + 1 & \text{if } \eta(j) = 0 \text{ and} \\ & w(j-1) = w(j), \\ \mu_0(\eta(j), w(j)) & \text{otherwise,} \end{cases}$$

where the sum is over all embeddings η of u in w .

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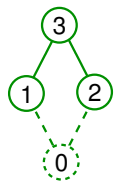
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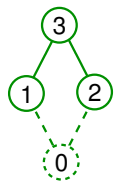
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A more extreme example. Calculate $\mu(\emptyset, 33333)$ when $P = \Lambda$.

The interval $[\emptyset, 33333]$ in P^* has 1906 edges!

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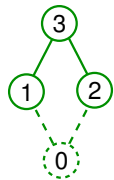
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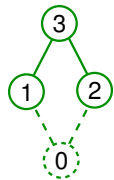
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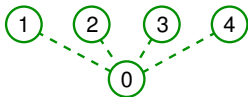
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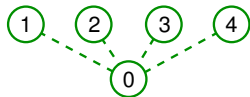
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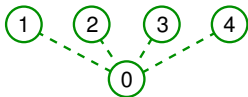
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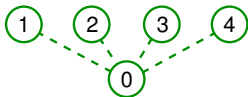
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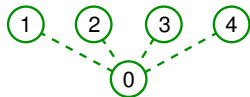
$$w = 23\color{red}{3}13$$

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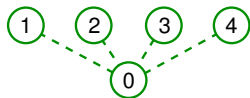
$$\eta = 20\color{red}003$$

$$(1)(-1)(-1+\color{red}1)(-1)(1)$$

“Normal embedding” (Björner): whenever $w(j-1) = w(j)$, need j th entry of embedding η to be nonzero.

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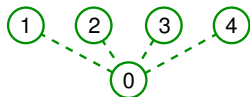
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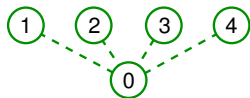
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$$\mu(u, w) = (-1)^{|w|-|u|} (\# \text{ normal embeddings}).$$

More applications

Application 2. Rederive Sagan & Vatter result for μ when P is a rooted forest.

Application 3. In particular, rederive Sagan & Vatter result for μ of composition order.

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Application 4. Rederive Tomie's result for $\mu(1^i, 3^j)$ when $P = \Lambda$.

$$\mu(1^i, 3^j) = [x^{j-i}]T_{i+j}(x) \quad \text{for } 0 \leq i \leq j$$

where $T_n(x)$ is the Chebyshev polynomial of the first kind.

$$T_0(x) = 1, \quad T_1(x) = x, \quad T_n(x) = 2xT_{n-1}(x) - T_{n-2}(x)$$

$$T_0(x) = 1$$

$$T_1(x) = x$$

$$T_2(x) = 2x^2 - 1$$

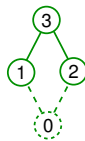
$$T_3(x) = 4x^3 - 3x$$

$$T_4(x) = 8x^4 - 8x^2 + 1$$

Tomie's result

Restate as

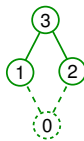
$$\mu(1^i, 3^j) = [x^{j-i}] T_{i+j}(x) = (-1)^i 2^{j-i-1} \frac{i+j}{j} \binom{j}{i}.$$



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Two types of embeddings of 1^i in 3^j :

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Contribution: $(-1)^i 2^{j-i}$

Number: $\binom{j-1}{i-1}$

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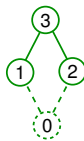
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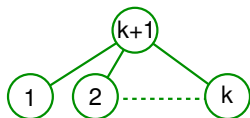
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$$\begin{aligned} \mu(1^i, 3^j) &= (-1)^i 2^{j-i-1} \left(2 \binom{j-1}{i-1} + \binom{j-1}{i} \right) \\ &= (-1)^i 2^{j-i-1} \left(\binom{j-1}{i-1} + \binom{j}{i} \right) \\ &= (-1)^i 2^{j-i-1} \frac{i+j}{j} \binom{j}{i}. \end{aligned}$$

Application 5. Tomie's results for augmented Λ .



Application 6. Suppose $\text{rk}(P) \leq 1$. Then any interval $[u, w]$ in P^* is

- ▶ shellable;
- ▶ homotopic to a wedge of $|\mu(u, w)|$ spheres, all of dimension $\text{rk}(w) - \text{rk}(u) - 2$.

Open problem. What if $\text{rk}(P) \geq 2$?

Summary so far

- ▶ Generalized subword order interpolates between subword order and an order on compositions.
- ▶ For any P , simple formula for the Möbius function of P^* in terms of Möbius function values of P .
- ▶ Formula implies all previously proved cases.

Method of proof

- ▶ Forman (1995): discrete Morse theory.
- ▶ Babson & Hersh (2005): discrete Morse theory for order complexes.
- ▶ Sagan & Vatter (2006): concise, accessible exposition.

Take-home message: if the usual methods for determining Möbius functions don't work, try DMT.

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In most of remaining time:
Mini-tutorial on using DMT to determine Möbius functions.

Method of proof

- ▶ Forman (1995): discrete Morse theory.
- ▶ Babson & Hersh (2005): discrete Morse theory for order complexes.
- ▶ Sagan & Vatter (2006): concise, accessible exposition.

Take-home message: if the usual methods for determining Möbius functions don't work, try DMT.

In most of remaining time:
Mini-tutorial on using DMT to determine Möbius functions.

We will **not** talk about

- ▶ CW-complexes,
- ▶ order complexes,
- ▶ Morse matchings,

and instead focus on the poset setting.

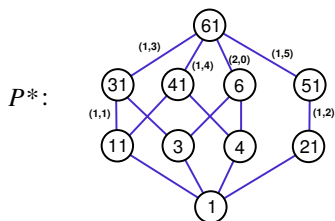
Step-by-step

Goal: compute $\mu(x, y)$ using DMT.

1. Pick \prec : an ordering of the maximal chains of $[x, y]$ that is a **poset lexicographic order** (PLO). *Note:* chains are read from top to bottom.
2. Identify the **skipped intervals (SIs)** of each maximal chain C : an interval I of the interior of C such that $C \setminus I \subseteq B$ for some maximal chain $B \prec C$.
3. Identify the **minimal skipped intervals (MSIs)** of C : the SIs that are minimal with respect to containment.
4. **Remove overlaps** among MSIs of C in a certain precise fashion to obtain the set $\mathcal{J}(C)$ of intervals.
5. If the $\mathcal{J}(C)$ cover the interior of C , then C is **critical**.
6. **Compute** the Möbius function:

$$\mu(x, y) = \sum_{\text{critical chains } C} (-1)^{|\mathcal{J}(C)|-1}.$$

Example from handout



$$61 \xrightarrow{(1,3)} 31 \xrightarrow{(1,1)} 11 \xrightarrow{(1,0)} 01$$

$$61 \xrightarrow{(1,3)} 31 \xrightarrow{(2,0)} 30 \xrightarrow{(1,1)} 10$$

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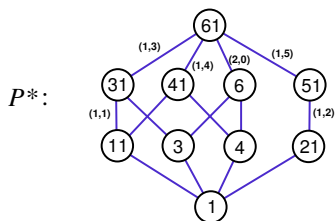
$$61 \xrightarrow{(1,4)} 41 \xrightarrow{(2,0)} 40 \xrightarrow{(1,1)} 10$$

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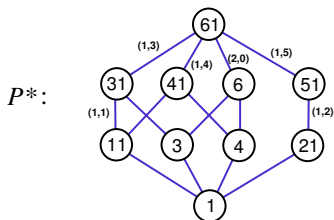
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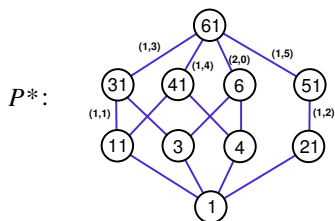
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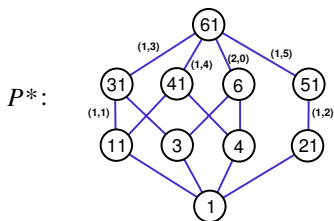
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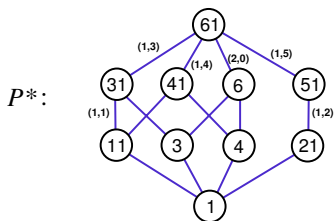
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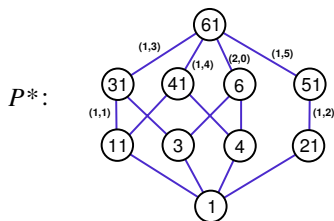
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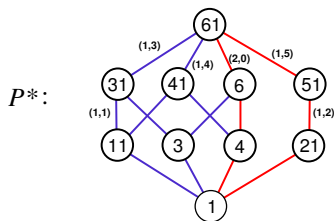
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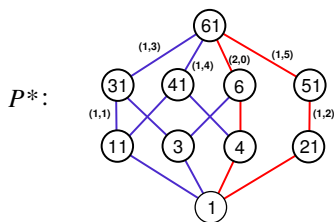
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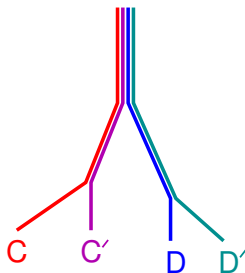
Poset lexicographic order

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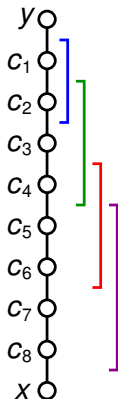
Answer: Whenever the setup in the picture occurs, we require that $C \prec D$ if and only if $C' \prec D'$.



Example. Start with an edge labeling with distinct “down labels” at any element. Then order the maximal chains lexicographically according to its edge labels.

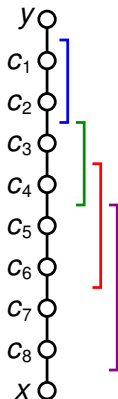
One last technicality...

What if MSIs overlap?



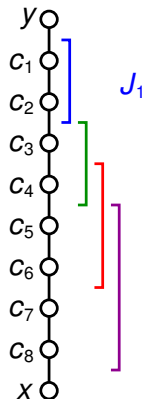
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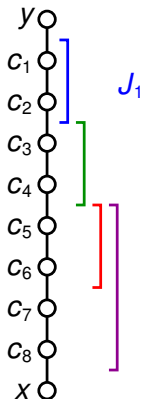
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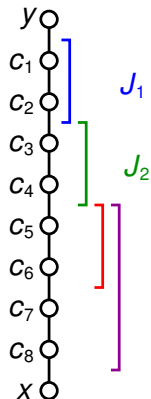
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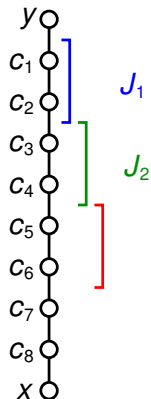
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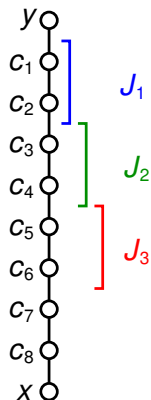
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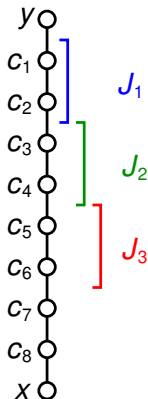
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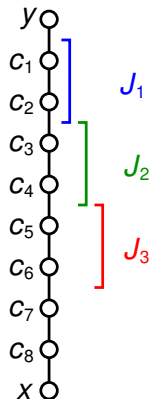
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This example is not a critical chain.

One last technicality...

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For the poset viewpoint of DMT, that's everything!
(great time for questions)

A word or two about the main proof

Theorem. Let P be a poset so that P_0 is locally finite. Let u and w be elements of P^* with $u \leq w$. Then

$$\mu(u, w) = \sum_{\eta} \prod_{1 \leq j \leq |w|} \begin{cases} \mu_0(\eta(j), w(j)) + 1 & \text{if } \eta(j) = 0 \text{ and} \\ & w(j-1) = w(j), \\ \mu_0(\eta(j), w(j)) & \text{otherwise,} \end{cases}$$

where the sum is over all embeddings η of u in w .

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2 key lemmas. Recall: edge labels are (position, element).

1. A descent in the position index creates a 1-element MSI.
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Upshot. When we reduce w to u along a critical chain:

1. we must work from right to left in w ;
2. every time we move left, we create an MSI that won't be involved in any MSI overlaps.

Building critical chains for generalized subword order

$\mu(1, 66)$ illustrates the key ideas.

Case 1: critical chains that end at embedding 01.

$$66 \xrightarrow{(2, \)} \xrightarrow{(2,1)} \boxed{61} \xrightarrow{(1, \)} \xrightarrow{(1,0)} 01$$

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13-page proof. Special treatment: $\mu(a0, ab)$ with $a < b$ in P .

Thanks \longrightarrow $\boxed{4}$ \longrightarrow listening!