Cylindric Skew Schur Functions

University of Minnesota Combinatorics Seminar

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Slides and preprint available from
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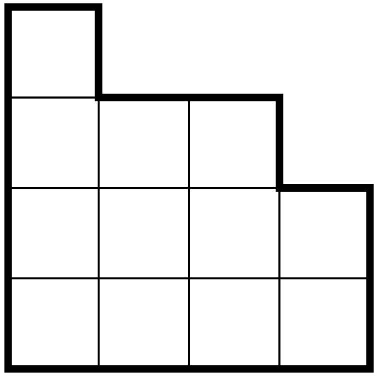
Schur functions

- Partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_l)$.
- Example: $(4, 4, 3, 1)$



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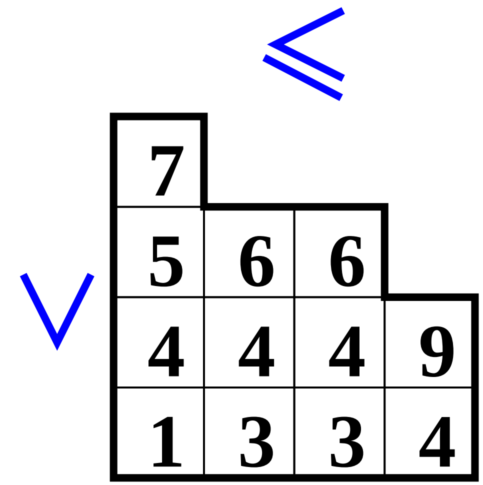
- Semistandard Young tableau (SSYT)



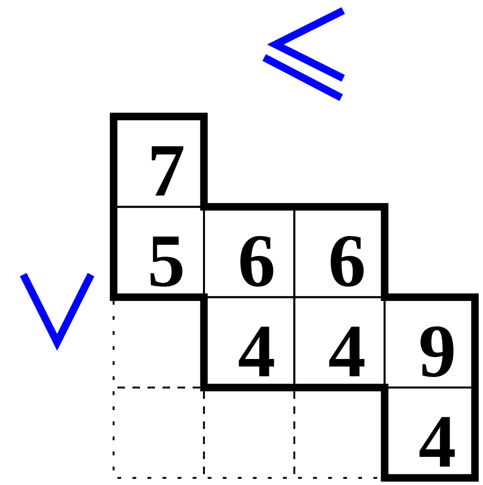
Schur function s_λ in the variables $\mathbf{x} = (x_1, x_2, \ldots)$ defined by

$$s_\lambda(\mathbf{x}) = \sum_{\text{SSYT } T} \mathbf{x}^T = \sum_{\text{SSYT } T} x_1^{\#1'\text{s in } T} x_2^{\#2'\text{s in } T} \cdots .$$

$$s_{4431}(\mathbf{x}) = x_1 x_3^2 x_4^4 x_5 x_6^2 x_7 x_9 + \cdots .$$

Skew Schur functions

- Partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_l)$.
- μ fits inside λ: form λ/μ.
- Example: $(4,4,3,1)/(3,1)$
- Semistandard Young tableau (SSYT)



Skew Schur function $s_{\lambda/\mu}$ in the variables $\mathbf{x} = (x_1, x_2, \ldots)$ defined by

$$s_{\lambda/\mu}(\mathbf{x}) = \sum_{\text{SSYT } T} \mathbf{x}^T = \sum_{\text{SSYT } T} x_1^{\#1'\text{s in } T} x_2^{\#2'\text{s in } T} \cdots .$$

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Do we care? For Schur!

- Schur functions are symmetric functions
- Schur functions s_λ form a basis for the symmetric functions.
- Arise in: representation theory of the symmetric group S_n.
- They are the characters of the irreducible representations of $\mathrm{GL}(n, \mathbb{C})$.
- Correspond to Schubert classes in $H^*(\mathrm{Gr}_{kn})$.

For skew Schur?

- Skew Schur functions are symmetric functions

$$s_{\lambda/\mu}(\mathbf{x}) = \sum_{\nu} c^{\nu}_{\lambda\mu} s_{\nu}(\mathbf{x}).$$

$c^{\nu}_{\lambda\mu}$: *Littlewood-Richardson coefficients*

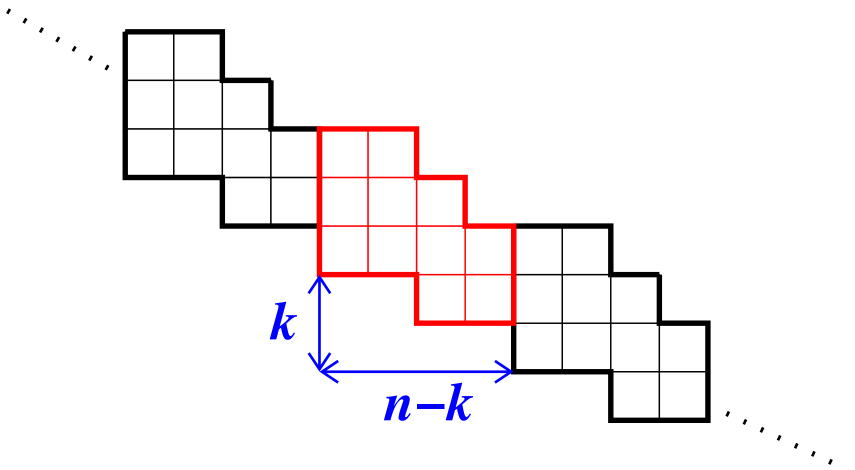
- Since $c^{\nu}_{\lambda\mu} \geq 0$, they are *Schur-positive*.

$$s_{4431/31} = s_{44} + 2s_{431} + s_{422} + s_{4211} + s_{332} + s_{3311}.$$

- Schur-positive symmetric functions are significant in the representation theory of S_n.

Cylindric skew Schur functions

- Infinite skew shape C
- Invariant under translation
- Identify (x, y) and $(x - n + k, y + k)$.



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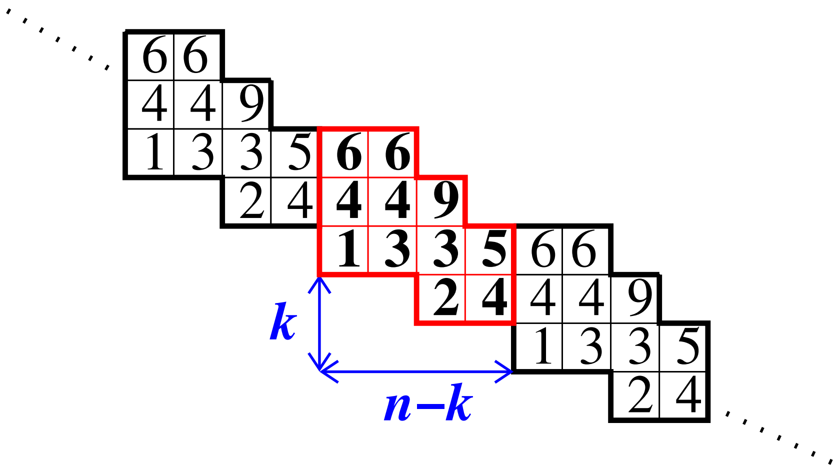


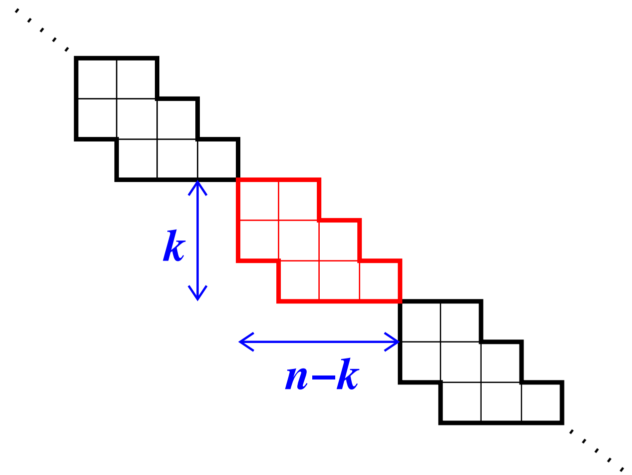
- Entries weakly increasing in each row
  Strictly increasing up each column
- Alternatively: SSYT with relations between entries in first and last columns

$$s_C(\mathbf{x}) = \sum_T \mathbf{x}^T = \sum_T x_1^{\#1'\text{s in } T} x_2^{\#2'\text{s in } T} \cdots .$$

- s_C is a symmetric function

Cylindric skew Schur functions

EXAMPLE



- Gessel,Krattenthaler: *“Cylindric Partitions”*
- Bertram, Ciocan-Fontanine, Fulton: *“Quantum Multiplication of Schur Polynomials”*
- Postnikov: *“Affine Approach to Quantum Schubert Calculus”* `math.CO/0205165`
- Stanley: *“Recent Developments in Algebraic Combinatorics”* `math.CO/0211114`

Motivation 1: P-partitions and an old conjecture of Stanley

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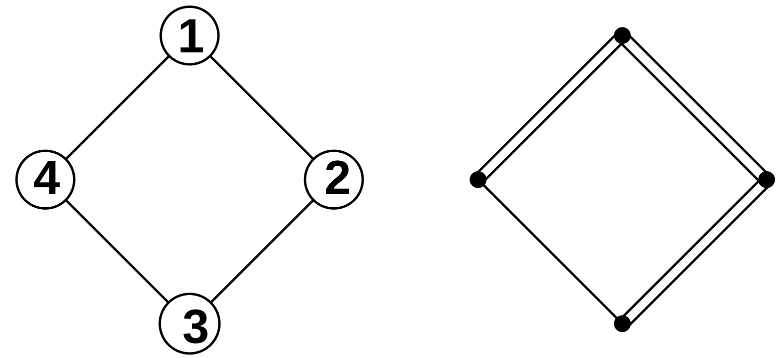
P: partially ordered set (poset)
$\omega : P \to \{1, 2, \ldots, |P|\}$ bijective labelling



DEFINITION (R. Stanley) Given a labelled poset (P, ω), a (P, ω)-partition is a map $f : P \to \mathbb{P}$ with the following properties:

- f is *order-preserving*: If $x \leq y$ in P then $f(x) \leq f(y)$
- If $x \lessdot y$ in P and $\omega(x) > \omega(y)$ then $f(x) < f(y)$

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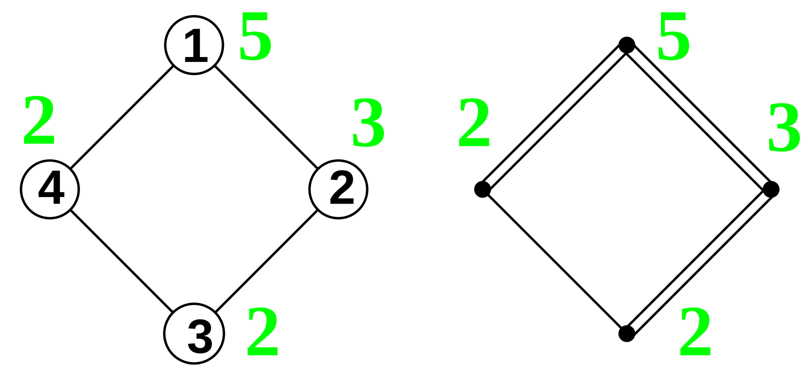

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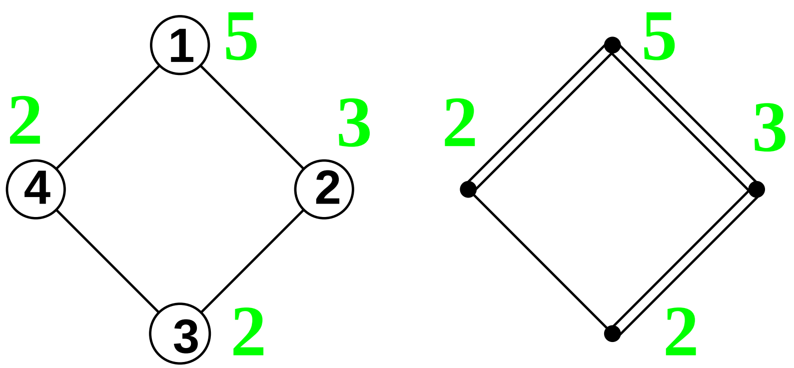

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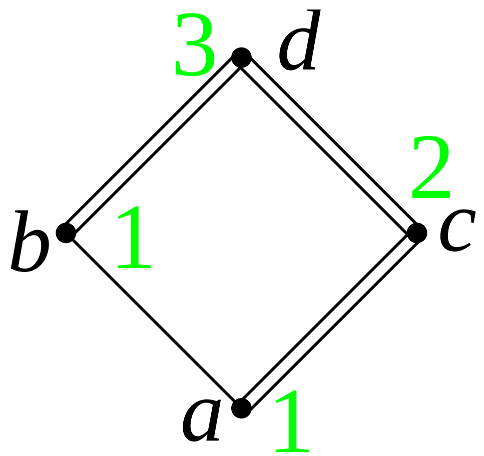
$$K_{P,\omega}(\mathbf{x}) = \sum_f \mathbf{x}^f = \sum_f x_1^{\#f^{-1}(1)} x_2^{\#f^{-1}(2)} \cdots .$$

A non-symmetric example

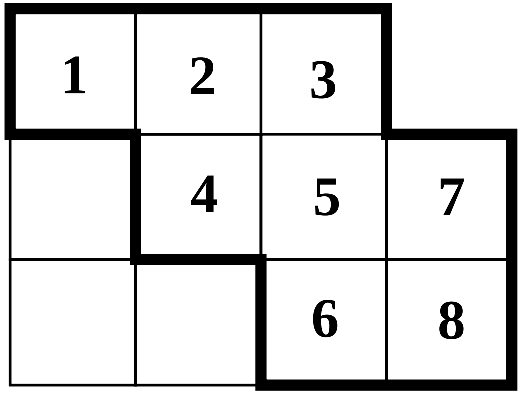
$$K_{P,\omega}(\mathbf{x}) = \sum_f \mathbf{x}^T = \sum_f x_1^{\#f^{-1}(1)} x_2^{\#f^{-1}(2)} \cdots .$$

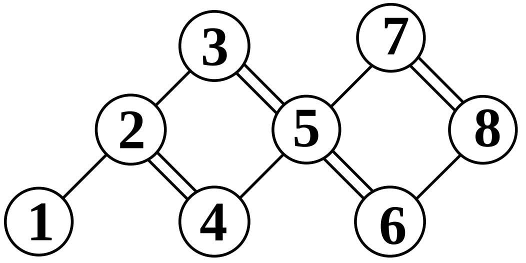
EXAMPLE



Coefficient of $x_1^2 x_2 x_3 = 1$
Coefficient of $x_1 x_2 x_3^2 = 0$
$\Rightarrow$ not symmetric

Schur labelled skew shape posets and Stanley's P-partitions Conjecture



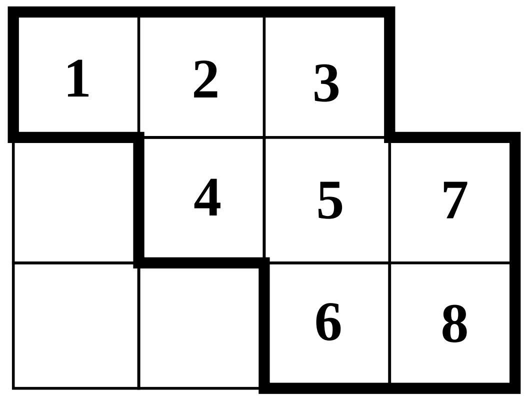


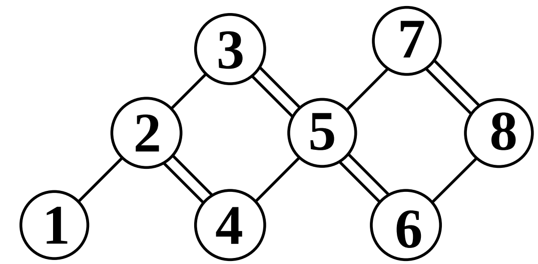
Bijection: SSYT of shape $\lambda/\mu \leftrightarrow (P,\omega)$-partitions
Furthermore,

$$K_{P,\omega}(\mathbf{x}) = s_{\lambda/\mu}(\mathbf{x}).$$

BIG QUESTION *What other labelled posets (P,ω) have symmetric $K_{P,\omega}(x)$?*

Schur labelled skew shape posets and Stanley's P-partitions Conjecture





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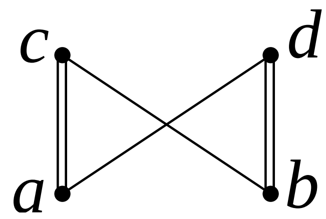
$$K_{P,\omega}(\mathbf{x}) = s_{\lambda/\mu}(\mathbf{x}).$$

BIG QUESTION *What other labelled posets (P,ω) have symmetric $K_{P,\omega}(x)$?*

CONJECTURE *(Stanley, c.1971) $K_{P,\omega}(\mathbf{x})$ is symmetric if and only if (P,ω) is isomorphic to a (Schur labelled) skew shape poset.*

Connection to cylindric skew Schur functions

EXAMPLE



We can check that $K_{P,\omega}(x)$ is symmetric.

So does it obey Stanley's conjecture?

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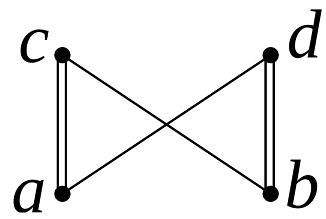

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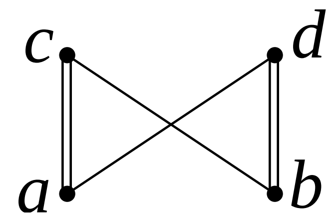

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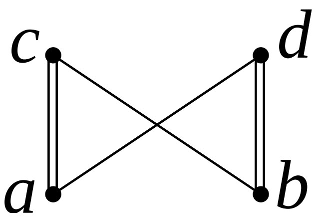

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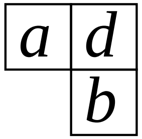

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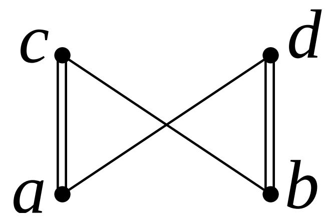

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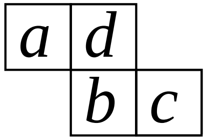

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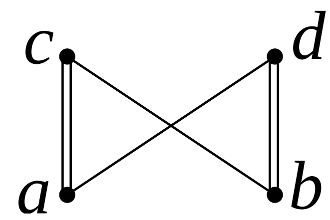

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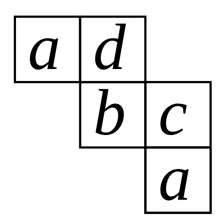

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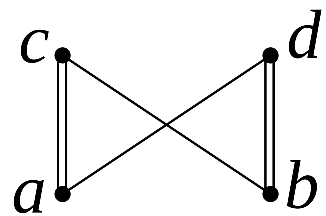

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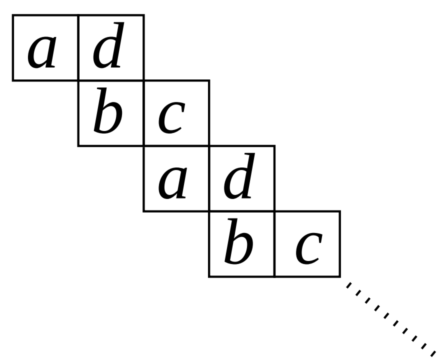

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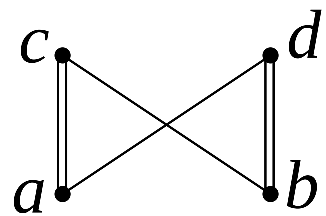

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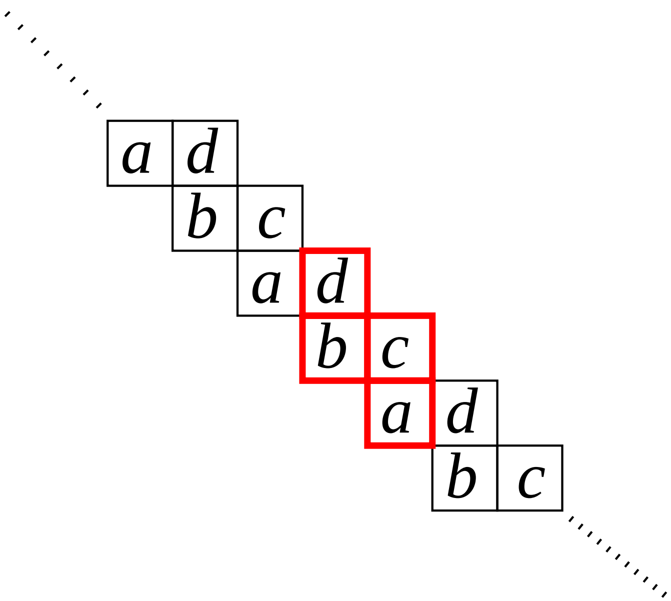

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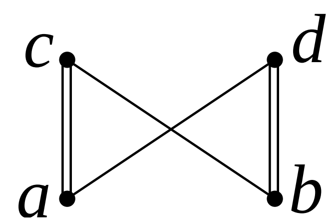

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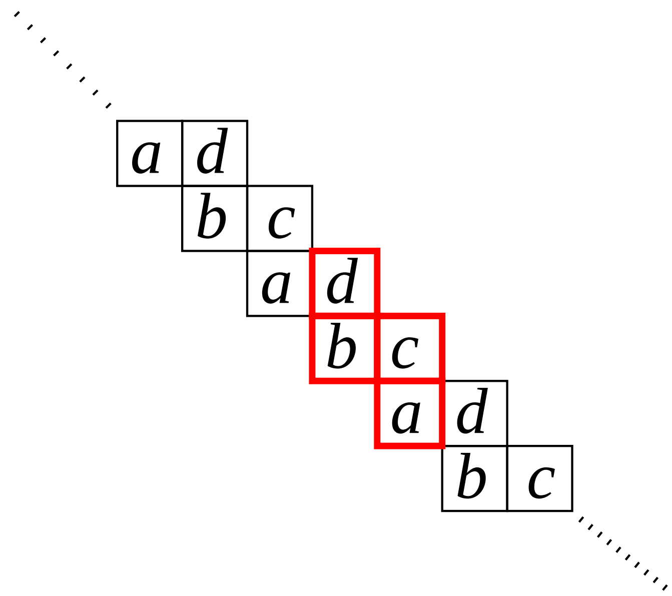

$\omega(a) > \omega(c) > \omega(b) > \omega(d) > \omega(a)$ Yikes!

Connection to cylindric skew Schur functions

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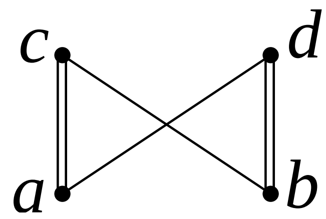

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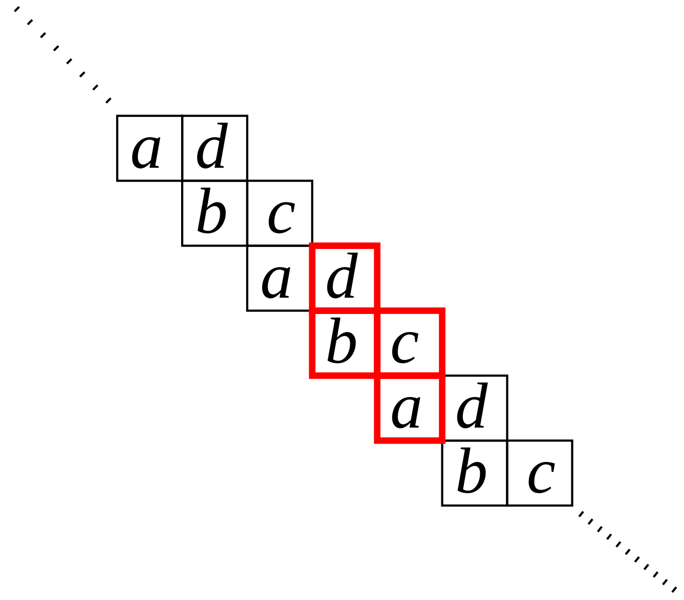
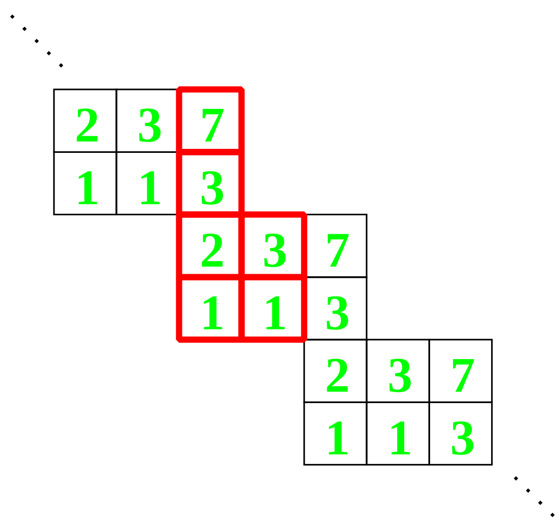
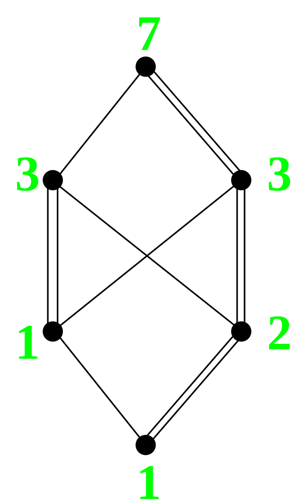


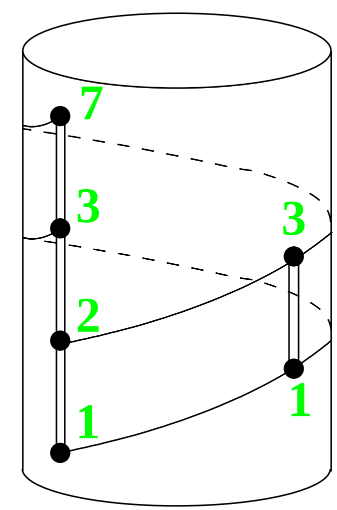
$\omega(a) > \omega(c) > \omega(b) > \omega(d) > \omega(a)$ Yikes! Oriented Poset

(P,O)-partitions

Labelled poset (P,ω)	Oriented poset (P,O)
$K_{P,\omega}(\mathbf{x})$	$K_{P,O}(\mathbf{x})$
skew shape posets	cylindric skew shape posets
skew Schur functions	cylindric skew Schur functions







Malvenuto's reformulation

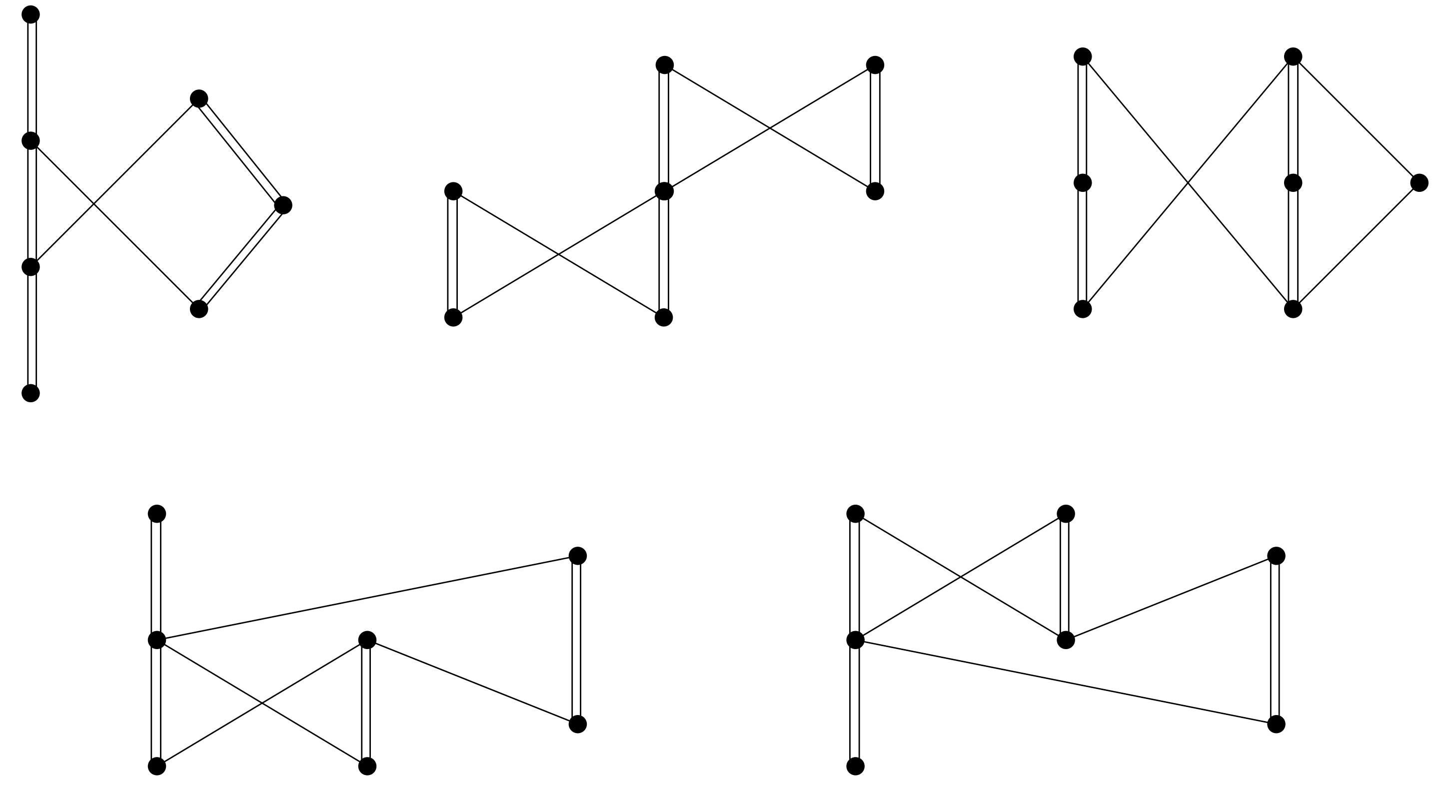
THEOREM *(C. Malvenuto, c. 1995) A labelled poset is a skew shape poset if and only if every connected component has no forbidden convex subposets*

THEOREM *(McN.) An oriented poset is a cylindric skew shape poset if and only if every connected component has no forbidden convex subposets*

CONJECTURE *(Stanley)* $K_{P,\omega}(\mathbf{x})$ *is symmetric if and only if every connected component of* (P,ω) *is isomorphic to a skew shape poset.*

CONJECTURE *(Stanley's conjecture extended to oriented posets)* $K_{P,O}(\mathbf{x})$ *is symmetric if and only if every connected component of* (P,O) *is isomorphic to a cylindric skew shape poset.*

Extended version is false!



Motivation 2: Positivity of Gromov-Witten invariants

In $H^*(Gr_{kn})$,

$$\sigma_\lambda \sigma_\mu = \sum_{\nu \subseteq k \times (n-k)} c_{\lambda\mu}^{\nu} \sigma_\nu .$$

In $QH^*(Gr_{kn})$,

$$\sigma_\lambda * \sigma_\mu = \sum_{d \geq 0} \sum_{\substack{\nu \vdash |\lambda| + |\mu| - dn \\ \nu \subseteq k \times (n-k)}} q^d C_{\lambda\mu}^{\nu,d} \sigma_\nu .$$

$C_{\lambda\mu}^{\nu,d} =$ 3-point Gromov-Witten invariants
$= \#\{$rational curves of degree d in Gr_{kn} that meet fixed generic translates of the Schubert varieties $\Omega_{\nu^\vee}$, Ω_λ and $\Omega_\mu\}$.

Key point: $C_{\lambda\mu}^{\nu,d} \geq 0$.

“Fundamental Open Problem”:

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Key point: $C^{\nu,d}_{\lambda\mu} \geq 0$.

"Fundamental Open Problem": Find an algebraic or combinatorial proof of this fact.

What's cylindric got to do with it?

THEOREM *(Postnikov)*

$$s_{\lambda/d/\mu}(x_1,\ldots,x_k)=\sum_{\nu\subseteq k\times(n-k)} C_{\lambda\mu}^{\nu,d}\, s_\nu(x_1,\ldots,x_k).$$

Conclusion: Want to understand expansions of cylindric skew Schur functions into Schur functions.

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THEOREM *(Postnikov)*

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Conclusion: Want to understand expansions of cylindric skew Schur functions into Schur functions.

COROLLARY $s_{\lambda/d/\mu}(x_1,x_2,\ldots,x_k)$ *is Schur-positive.*

Known: $s_{\lambda/d/\mu}(x_1,x_2,\ldots)$ need not be Schur-positive.

Note: $s_\lambda(x_1,x_2,\ldots,x_k)\neq 0 \Leftrightarrow \lambda$ has at most k rows.

Example: If $s_{\lambda/d/\mu} = s_{22}+s_{211}-s_{1111}$, then $s_{\lambda/d/\mu}(x_1,x_2,x_3) = s_{22}+s_{211}$ is Schur-positive.

When is a cylindric skew Schur function Schur-positive?

THEOREM *(McN.) For any cylindric skew shape C,*

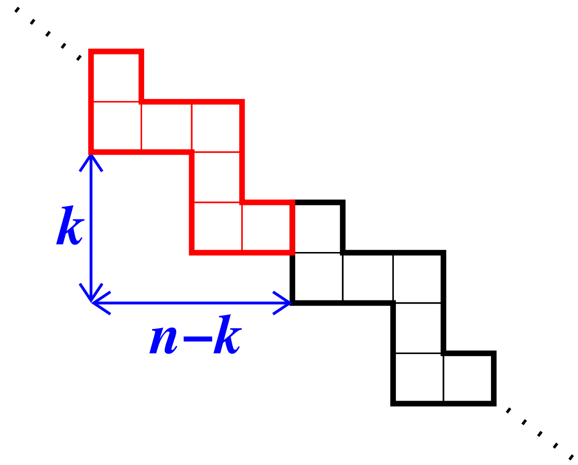
$s_C(x_1, x_2, \ldots)$ *is Schur-positive* $\Leftrightarrow$ C *is a skew shape*.

Equivalently, if C is a non-trivial cylindric skew shape, then $s_C(x_1, x_2, \ldots)$ is not Schur-positive.

Example: Cylindric ribbons

EXAMPLE

C:



$$s_C(x_1, x_2, \ldots) \;=\; \sum_{\nu \subseteq k\times(n-k)} c_\nu s_\nu \;+ s_{n-k,1^k} - s_{n-k-1,1^{k+1}}$$

$$+ s_{n-k-2,1^{k+2}} - \cdots + (-1)^{n-k} s_{1^n}.$$

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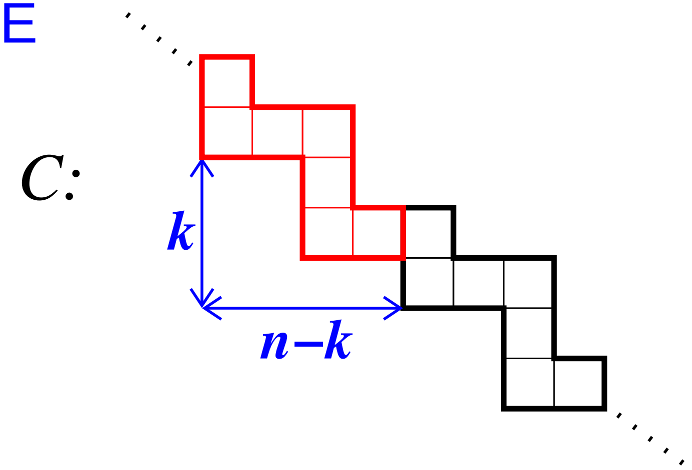

$$s_C(x_1, x_2, \ldots) = \sum_{\nu \subseteq k \times (n-k)} c_\nu s_\nu + s_{n-k,1^k} - s_{n-k-1,1^{k+1}} + s_{n-k-2,1^{k+2}} - \cdots + (-1)^{n-k} s_{1^n}.$$

Schur-positive with $k+1$ variables

Not Schur-positive with $\geq k+2$ variables

General cylindric skew shape: $\geq k+2+l$ variables

Shapes in Postnikov's theorem: $\geq 2k+1$ variables

Tool: Cylindric skew Schur functions as signed sums of skew Schurs

Bertram, Ciocan-Fontanine, Fulton:

☺ Nice description in terms of ribbons

☹ Only for certain shapes, certain terms

Gessel, Krattenthaler:

☺ Works for all cylindric skew shapes

☹ Not as nice a description

We can get the best of both worlds:
A technique for expanding a cylindric skew Schur function in terms of skew Schur functions that
Works for all cylindric skew shapes like G-K and has a nice description like B-CF-F

Formula of Bertram, Ciocan-Fontanine, Fulton

THEOREM *(B-CF-F) For λ, μ, $\nu \subseteq k \times (n-k)$ with $|\mu| + |\nu| = |\lambda| + dn$ for some $d \geq 0$, we have*

$$C^{\lambda,d}_{\mu\nu} = \sum_{\tau} \varepsilon(\tau/\lambda) c^{\tau}_{\mu\nu}$$

where the sum is over all τ with $\tau_1 \leq n-k$ that can be obtained from λ by adding d n-ribbons.

Formula of Bertram, Ciocan-Fontanine, Fulton

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$$\sum_{v} C_{\mu\nu}^{\lambda,d} s_\nu(x_1, \ldots, x_k) = \sum_{v} \sum_{\tau} \varepsilon(\tau/\lambda) c_{\mu\nu}^{\tau} s_\nu(x_1, \ldots, x_k)$$

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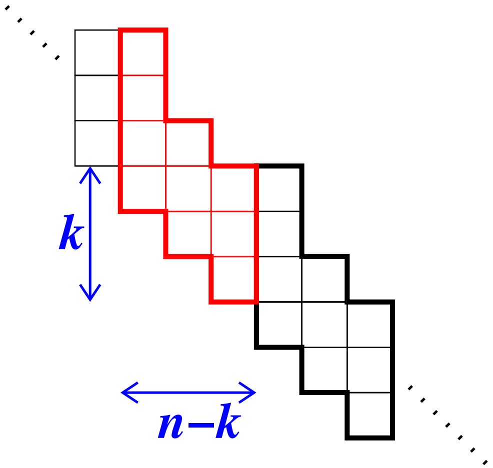
COROLLARY *For any cylindric skew shape* $\lambda/d/\mu$ *with* $\lambda, \mu \subseteq k \times (n-k)$, *we have*

$$s_{\lambda/d/\mu}(x_1, \ldots, x_k) = \sum_{\tau} \varepsilon(\tau/\lambda) s_{\tau/\mu}(x_1, \ldots, x_k),$$

where the sum is over all τ *with* $\tau_1 \leq n-k$ *that can be obtained from* λ *by adding* d n*-ribbons.*

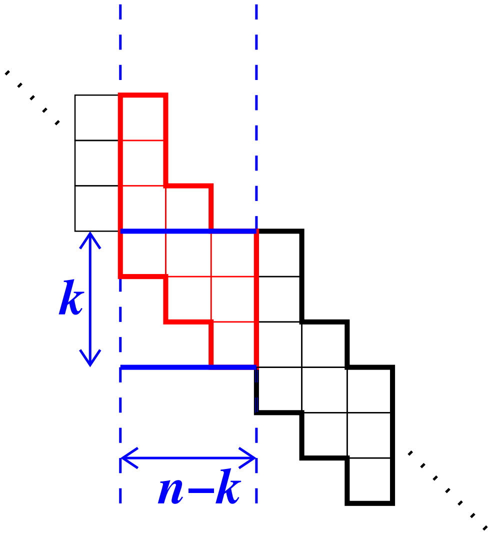
Tool: Cylindric skew Schur functions as signed sums of skew Schurs

EXAMPLE



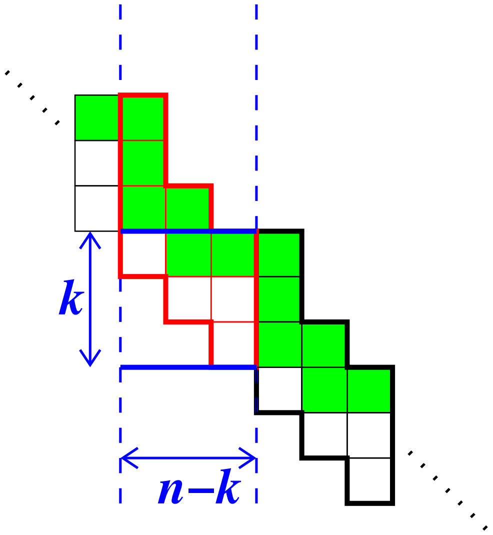
Tool: Cylindric skew Schur functions as signed sums of skew Schurs

EXAMPLE



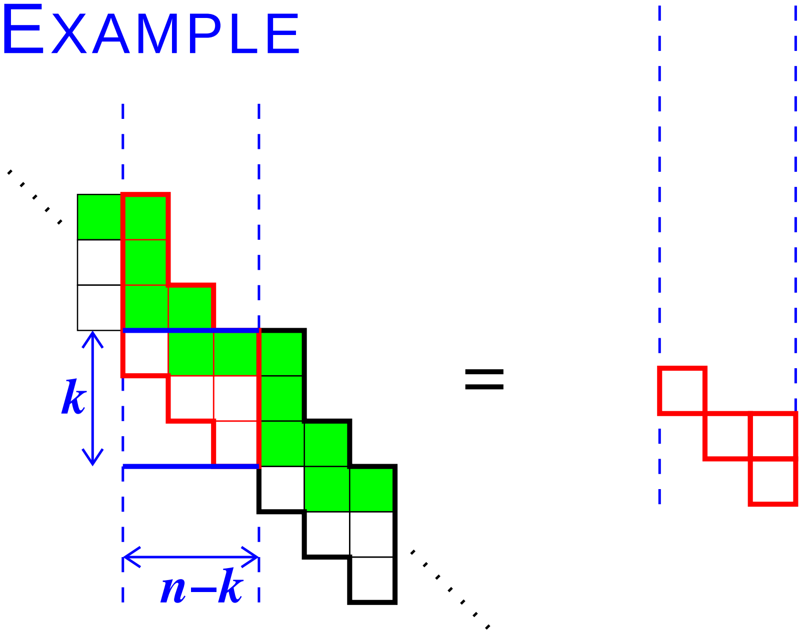
Tool: Cylindric skew Schur functions as signed sums of skew Schurs

EXAMPLE



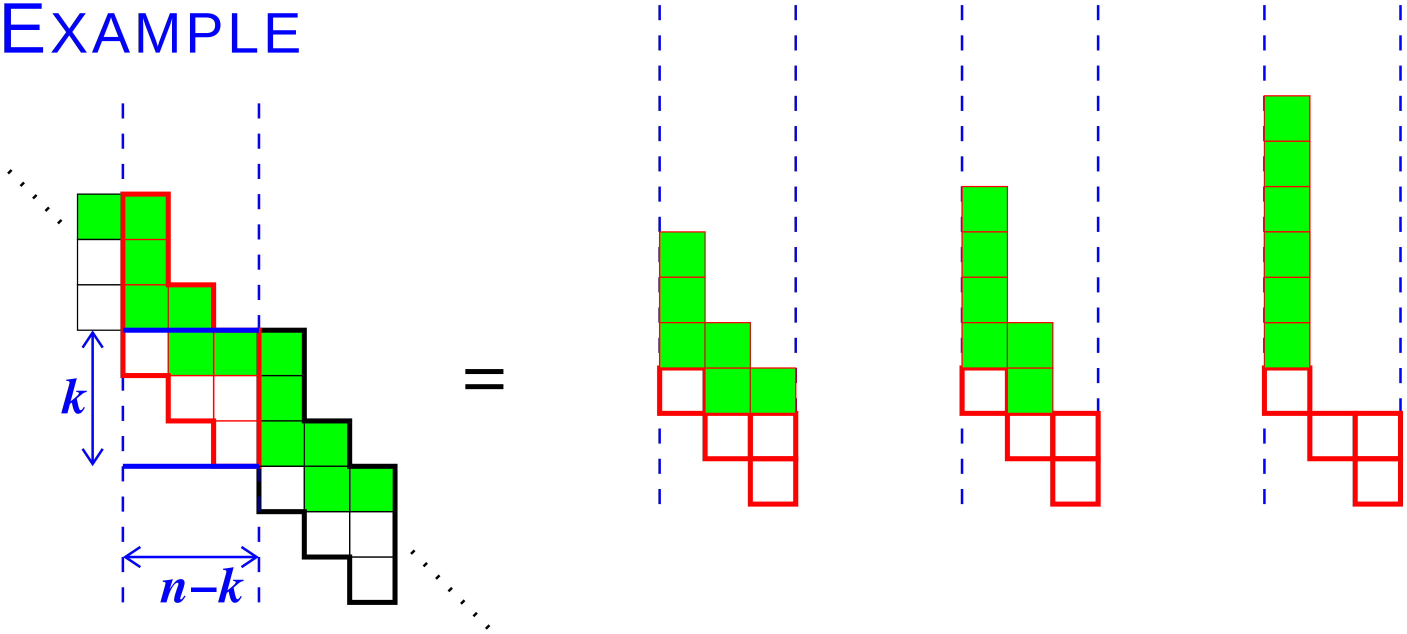
Tool: Cylindric skew Schur functions as signed sums of skew Schurs

Example



Tool: Cylindric skew Schur functions as signed sums of skew Schurs

EXAMPLE



Tool: Cylindric skew Schur functions as signed sums of skew Schurs

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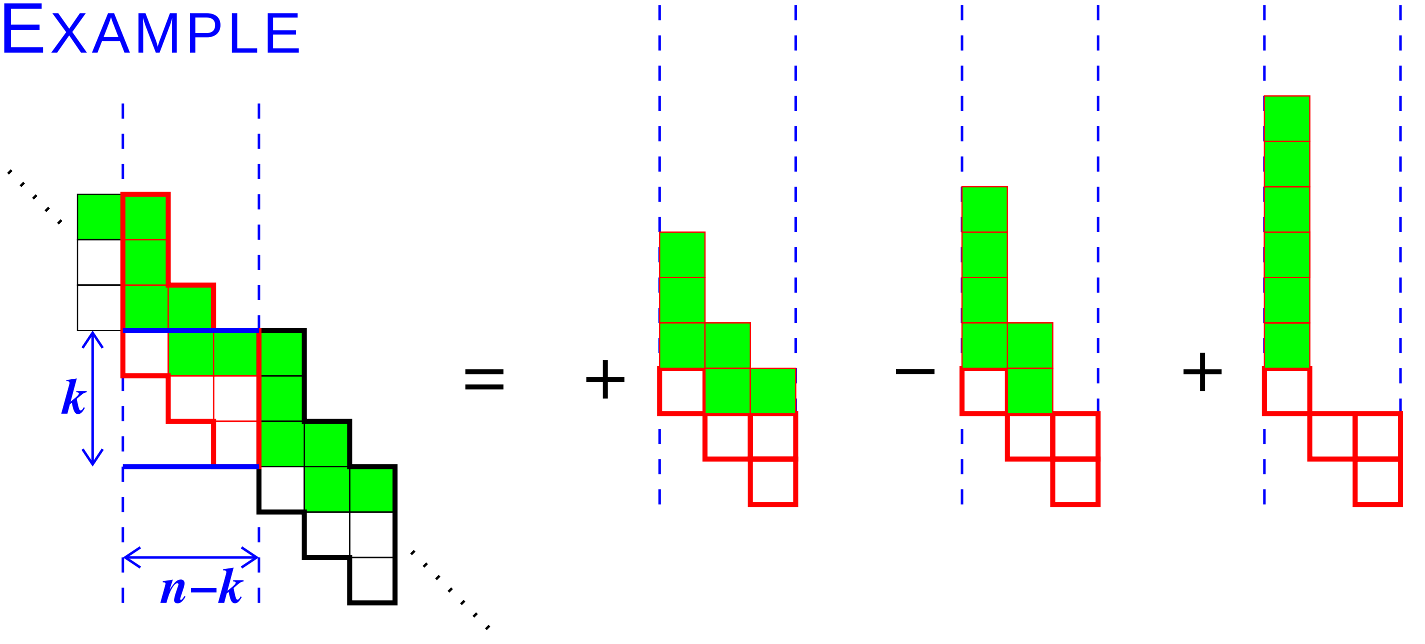
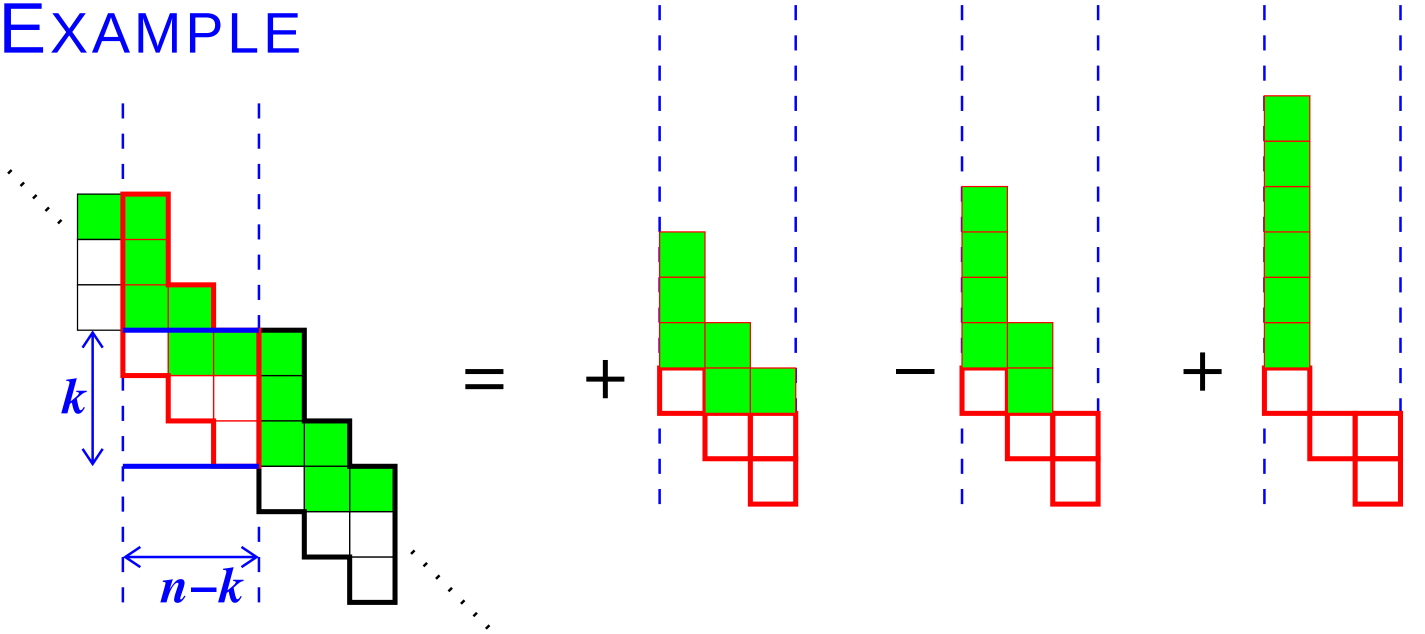


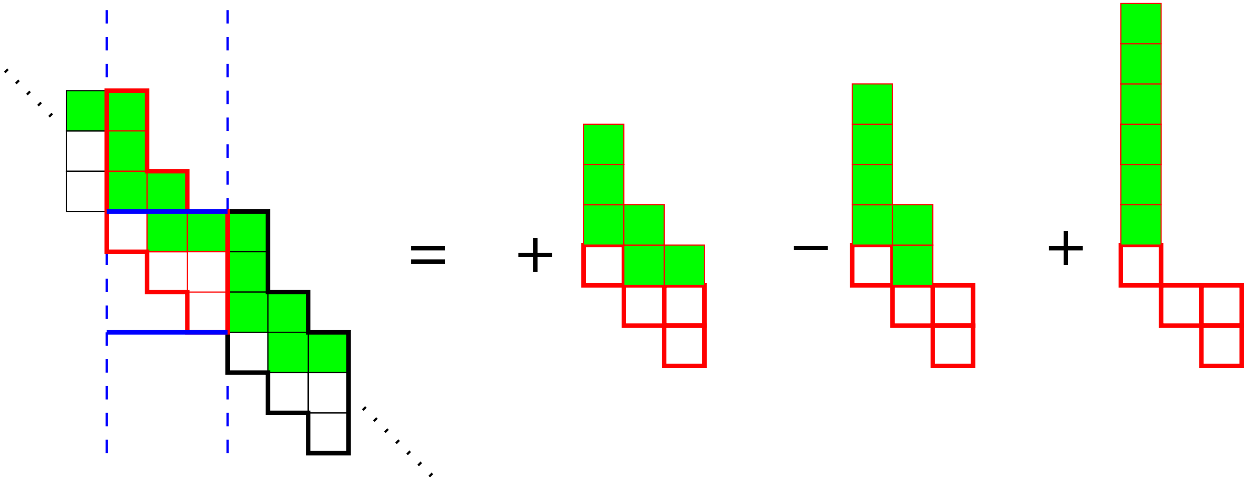
Tool: Cylindric skew Schur functions as signed sums of skew Schurs

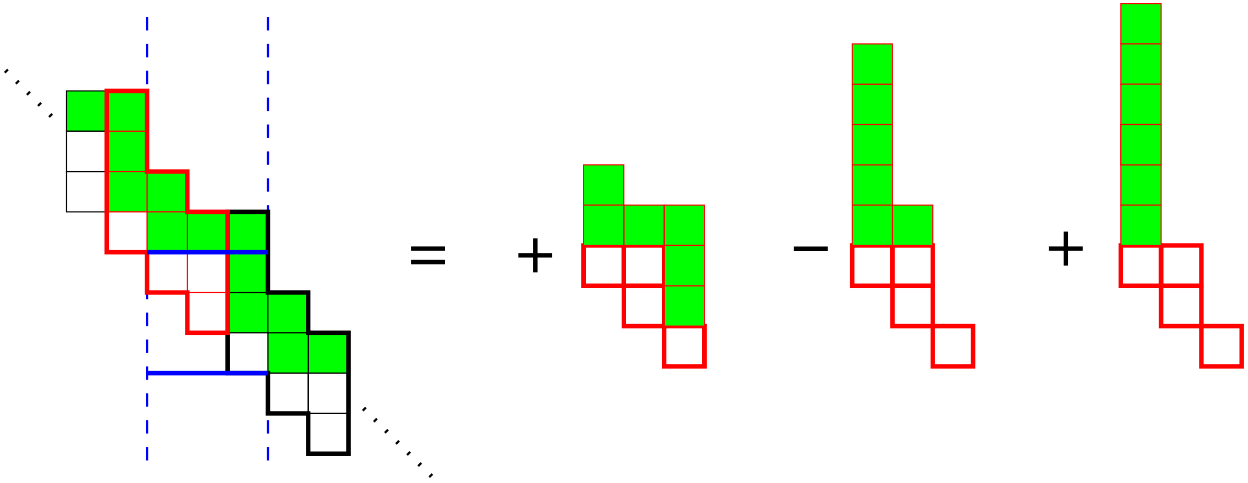
EXAMPLE



$$
\begin{aligned}
s_C &= s_{333211/21} - s_{3322111/21} + s_{331111111/21} \\
&= s_{3331} + s_{3322} + s_{33211} + s_{322111} + s_{31111111} \\
&\quad - s_{222211} - s_{2221111} + s_{22111111} + s_{211111111}.
\end{aligned}
$$

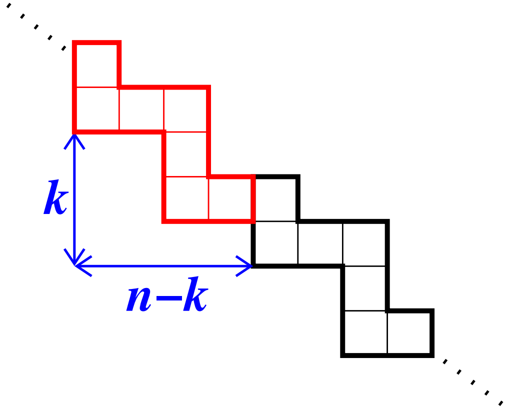
Consequence: Lots of skew Schur function identities





Example: Cylindric ribbons

C:



$$s_C(x_1, x_2, \ldots) = \sum_{\nu \subseteq k \times (n-k)} c_\nu s_\nu + s_{n-k,1^k} - s_{n-k-1,1^{k+1}}$$
$$+ s_{n-k-2,1^{k+2}} - \cdots + (-1)^{n-k} s_{1^n}.$$

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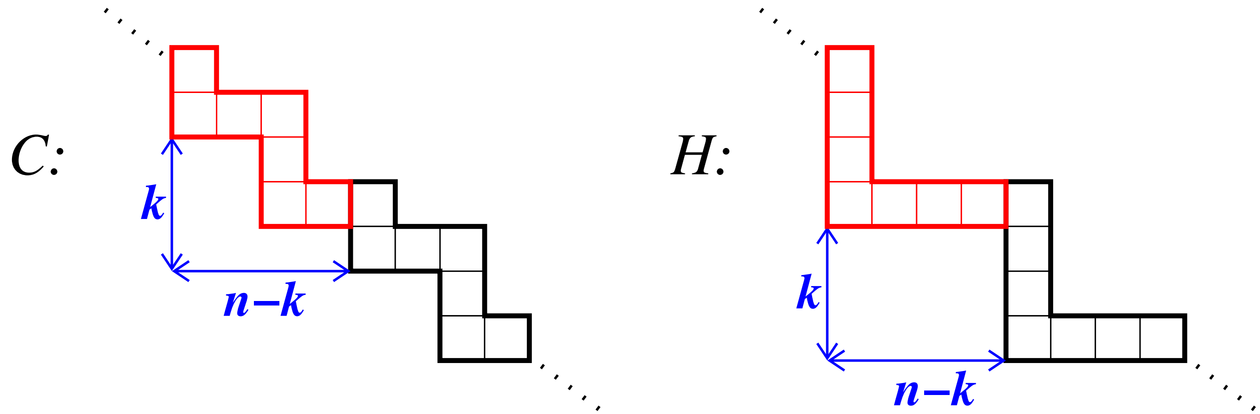

$$s_C(x_1, x_2, \ldots) = \sum_{\nu \subseteq k \times (n-k)} c_\nu s_\nu + s_{n-k,1^k} - s_{n-k-1,1^{k+1}}$$
$$+ s_{n-k-2,1^{k+2}} - \cdots + (-1)^{n-k} s_{1^n}.$$

However, $s_C(x_1, x_2, \ldots) = \sum_{\nu \subseteq k \times (n-k)} c_\nu s_\nu + s_H.$

s_C: cylindric *skew* Schur function

s_H: cylindric Schur function

We say that s_C is cylindric Schur-positive.

A Conjecture

CONJECTURE *For any cylindric skew shape C, s_C is cylindric Schur-positive.*

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THEOREM *(McN.) The cylindric Schur functions corresponding to a given translation $(-n+k,+k)$ are linearly independent.*

THEOREM *(McN.) If s_C can be written as a linear combination of cylindric Schur functions with the same translation as C, then s_C is cylindric Schur-positive.*