When are Two Skew Schur Functions Equal?

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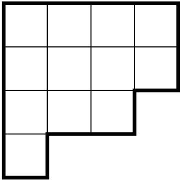
Slides and paper available from
`www.facstaff.bucknell.edu/pm040/`

Outline

- Background: skew Schur functions
- Recent work on skew Schur functions
- Skew Schur equivalence
- Composition of skew diagrams, main results
- Conjectures, open problems

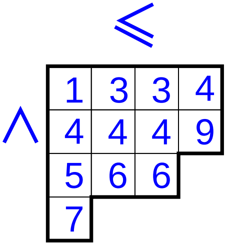
- Partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_\ell)$

- Young diagram.
  Example:
  $\lambda = (4, 4, 3, 1)$



Schur functions

- Partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_\ell)$
- Young diagram.
  Example:
  $\lambda \quad = (4, 4, 3, 1)$
- Semistandard Young tableau (SSYT)



The Schur function s_λ in the variables $x = (x_1, x_2, \ldots)$ is then defined by

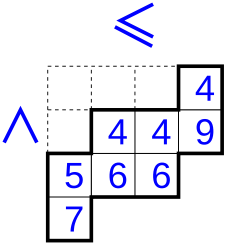
$$s_\lambda = \sum_{\text{SSYT } T} x_1^{\#\text{1's in } T} x_2^{\#\text{2's in } T} \cdots .$$

Example

$$s_{4431} = x_1 x_3^2 x_4^4 x_5 x_6^2 x_7 x_9 + \cdots .$$

Skew Schur functions

- Partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_\ell)$
- μ fits inside λ.
- Young diagram.
  Example:
  $\lambda/\mu = (4,4,3,1)/(3,1)$
- Semistandard Young tableau (SSYT)



The skew Schur function $s_{\lambda/\mu}$ in the variables $x = (x_1, x_2, \ldots)$ is then defined by

$$s_{\lambda/\mu} = \sum_{\text{SSYT } T} x_1^{\#\text{1's in } T} x_2^{\#\text{2's in } T} \cdots .$$

Example

$$s_{4431/31} = \qquad x_4^3 x_5 x_6^2 x_7 x_9 + \cdots .$$

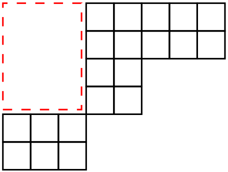
Notes on skew Schur functions

- Skew Schur functions are symmetric in the variables $x = (x_1, x_2, \ldots)$.
- The Schur functions form a basis for the algebra of symmetric functions (over $\mathbb{Q}$, say).
- Connections with Algebraic Geometry, Representation Theory
- Skew Schur functions are *Schur-positive*:

$$s_{\lambda/\mu} = \sum_{\nu} c^{\lambda}_{\mu\nu} s_{\nu} \text{ where } c^{\lambda}_{\mu\nu} \geq 0.$$

The study of skew Schur functions is a hot area

- John Stembridge (2000): Complete classification of multiplicity-free products of Schur functions:



- Christian Gutschwager (August 2006): Complete classification of multiplicity-free skew Schur functions.

The study of skew Schur functions is a hot area

We know that $s_\lambda s_\mu$ is Schur-positive. When is $s_\sigma s_\tau - s_\lambda s_\mu$ Schur-positive?

- Sergey Fomin, William Fulton, Chi-Kwong Li, Yiu-Tung Poon (2003):
  2 conjectures for operations.
  First operation: $(\lambda, \mu) \to (\tilde{\lambda}, \tilde{\mu})$.
  Conjecture: $s_{\tilde{\lambda}} s_{\tilde{\mu}} - s_\lambda s_\mu$ is Schur-positive.
- Anatol Kirillov, François Bergeron, McN. (2004):
  Conjecture: $s_{\tilde{\lambda}/\tilde{\alpha}} s_{\tilde{\mu}/\tilde{\beta}} - s_{\lambda/\alpha} s_{\mu/\beta}$ is Schur-positive.
- Thomas Lam, Alexander Postnikov, Pavlo Pylyavskyy (2005):
  Proof of conjectures.
- Second FFLP operation: $(\lambda, \mu) \to (\lambda^*, \mu^*)$.
  François Bergeron, Riccardo Biagioli, Mercedes Rosas (2004):
  partial results.

The HDL series

When is $s_{\lambda/\alpha} = s_{\mu/\beta}$?

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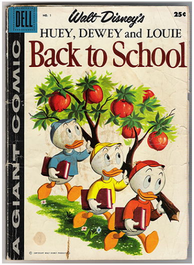

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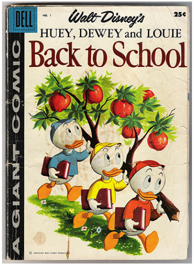
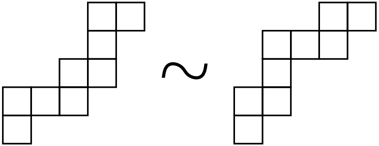


Complete classification of equality of ribbon Schur functions



The HDL series

- HDL II: Vic Reiner, Kristin Shaw, Steph van Willigenburg (2006):
  - The more general setting of binomial syzygies

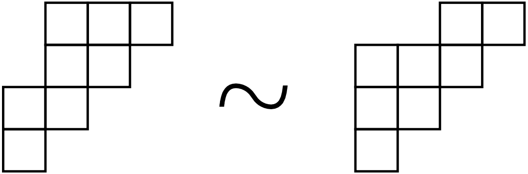
    $$c s_{D_1} s_{D_2} \cdots s_{D_m} = c' s_{D'_1} s_{D'_2} \cdots s_{D'_n}$$

    is equivalent to understanding equalities among connected skew diagrams.
  - 3 operations for generating skew diagrams with equal skew Schur functions.
  - Necessary conditions, but of a different flavor.

- HDL III: McN., Steph van Willigenburg (2006):
  - An operation that encompasses the three operations of HDL II.
  - Theorem that generalizes all previous results.
    Explains the 6 missing equivalences from HDL II.
  - Conjecture for necessary and sufficient conditions for $s_{\lambda/\alpha} = s_{\mu/\beta}$.
    Reflects classification of HDL I for ribbons.

Skew diagrams (skew shapes) D, E.
If $s_D = s_E$, we will write $D \sim E$.

Example



We want to classify all equivalences classes, thereby classifying all skew Schur functions.

The basic building block

EC2, Exercise 7.56(a) [2-]

Theorem

$D \sim D^*$, where D^* denotes D rotated by 180°.

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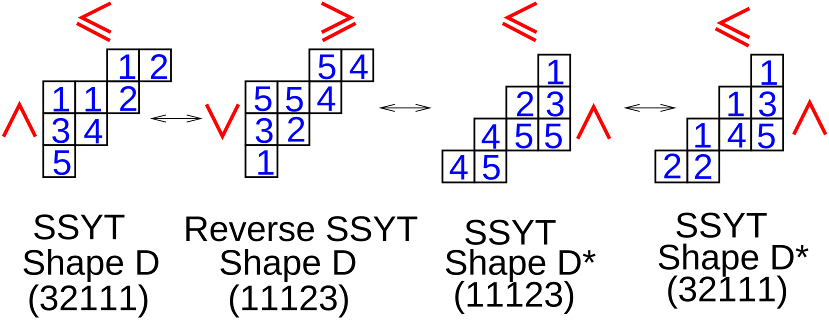
Proof.



□

Goal: Use this equivalence to build other skew equivalences.

Where we're headed

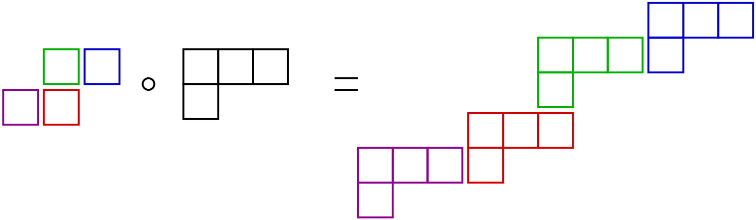
Theorem

Suppose we have skew diagrams D, D' and E satisfying certain assumptions. If $D \sim D'$ then

$$D' \circ_W E \quad \sim \quad D \circ_W E \quad \sim \quad D \circ_{W^*} E^*.$$

Main definition: composition of skew diagrams.

Composition of skew diagams



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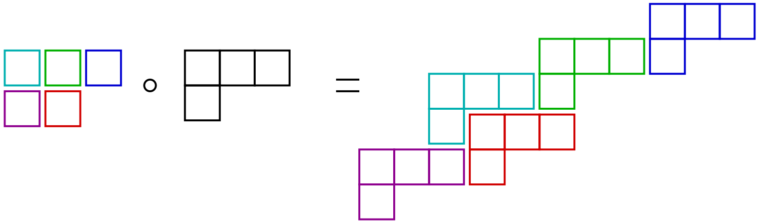

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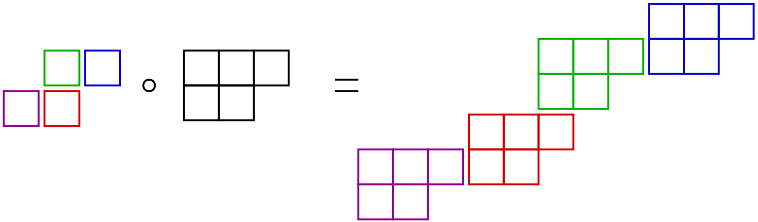

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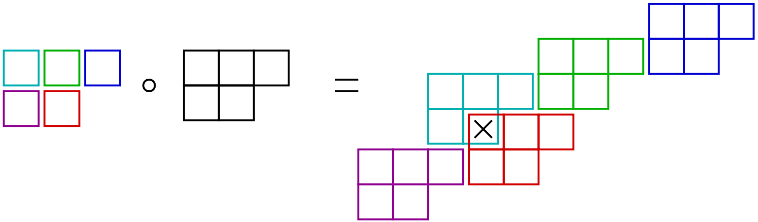

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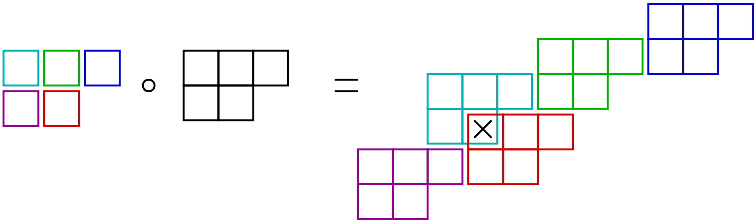
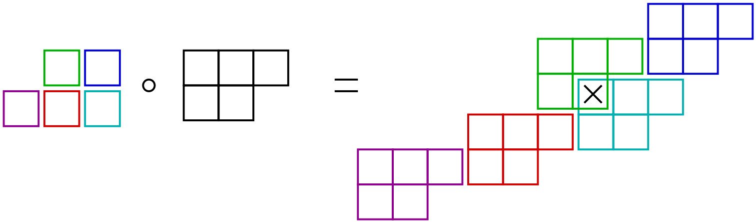


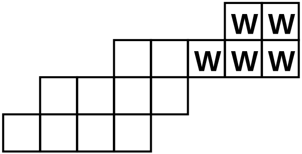
Theorem [McN., van Willigenburg] *If* $D \sim D'$, *then*

$$D' \circ E \quad \sim \quad D \circ E \quad \sim \quad D \circ E^*.$$



Actually, the previous slide was just a warm-up....

A skew diagram W *lies in the top* of a skew diagram E if W appears as a connected subdiagram of E that includes the northeasternmost cell of E.



Amalgamated Compositions

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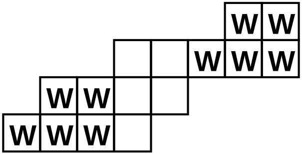

Similarly, W *lies in the bottom* of E.

Our interest: W lies in both the top and bottom of E. We write $E = WOW$.

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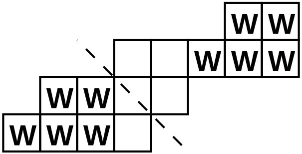


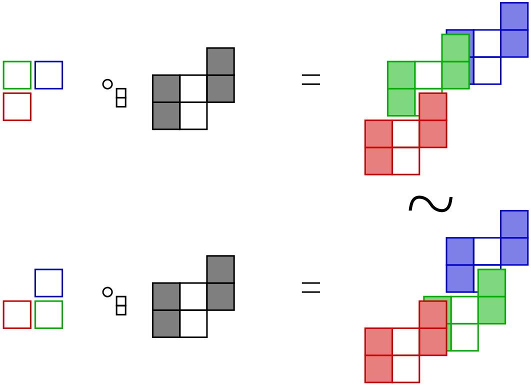
Similarly, W *lies in the bottom* of E.

Our interest: W lies in both the top and bottom of E. We write $E = WOW$.

Hypotheses: (inspired by hypotheses of RSvW)

1. W is maximal given its set of diagonals.
2. W_{ne} and W_{sw} are separated by at least one diagonal.
3. $E \setminus W_{ne}$ and $E \setminus W_{sw}$ are both connected skew diagrams.

Amalgamated Compositions



15 boxes: first of the non-RSvW examples

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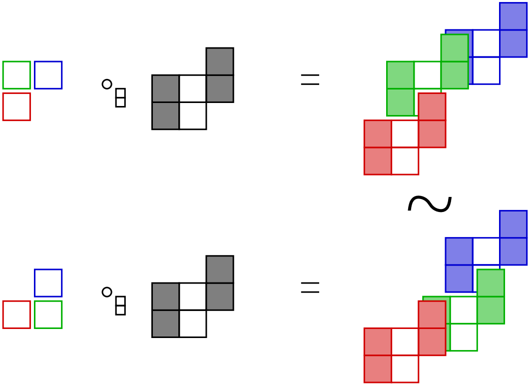
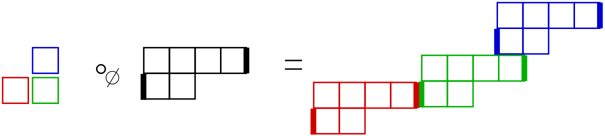


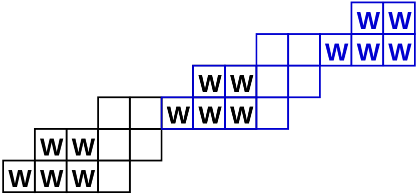
15 boxes: first of the non-RSvW examples

If $W = \emptyset$, we get the regular compositions:



What are the results?

Construction of $\overline{W}$ and $\overline{O}$:



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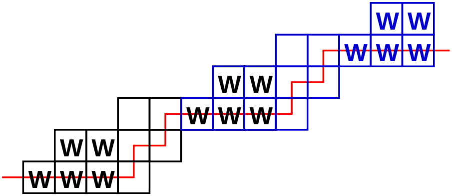

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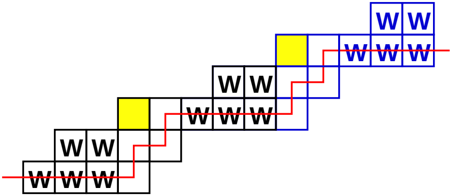

Hypothesis 4. $\overline{W}$ is never adjacent to $\overline{O}$.

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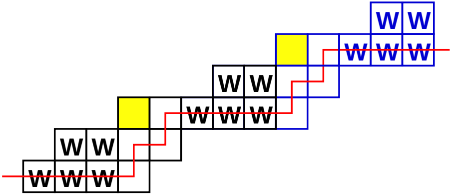

Hypothesis 4. $\overline{W}$ is never adjacent to $\overline{O}$.

Conjecture. Suppose we have skew diagrams D, D' with $D \sim D'$ and $E = WOW$ satisfying Hypotheses 1-4, then

$$D' \circ_W E \sim D \circ_W E \sim D \circ_{W^*} E^*.$$

What are the results?

Hypothesis 5. In $E = WOW$, at least one copy of W has just one cell adjacent to O.

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Theorem.[McN., van Willigenburg] Suppose we have skew diagrams D, D' with $D \sim D'$ and $E = WOW$ satisfying Hypotheses 1-5, then

$$D' \circ_W E \sim D \circ_W E \sim D \circ_{W^*} E^*.$$

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Theorem.[McN., van Willigenburg] Suppose we have skew diagrams D, D' with $D \sim D'$ and $E = WOW$ satisfying Hypotheses 1-5, then

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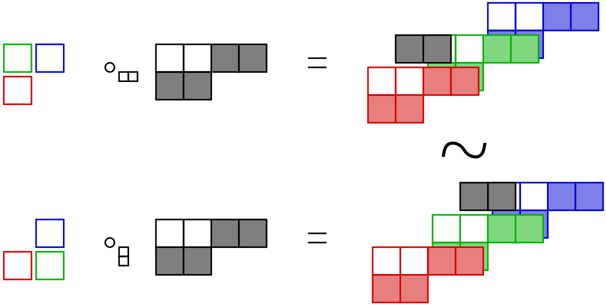


15 boxes: second of the non-RSvW examples

A word or two about the proof

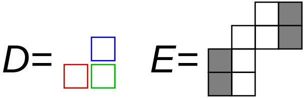
The hard part: An expression for $s_{D \circ_W E}$ in terms of s_D, s_E, $s_{\overline{W}}$, $s_{\overline{O}}$.

The easy part: The expression is invariant if we replace D by D' when $D' \sim D$. Similary, can replace E by E^*.

Proof of expression uses:

- Hamel-Goulden determinants. See paper of Chen, Yan, Yang.
- Sylvester's Determinantal Identity.

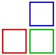
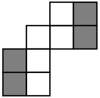
- Removing Hypothesis 5.



$D \circ_W E$ has 23 boxes.

Open problems

- Removing Hypothesis 5.

$D=$  $E=$ 

$D \circ_W E$ has 23 boxes.

- When is $F \sim F^t$?

Know: If $E^t = E$ and $W^t = W$, then $(D \circ_W E)^t \sim D \circ_W E$.

Conjecture: This is the only way that $F \sim F^t$.

i.e. If $F \sim F^t$ with $F \neq F^t$, then there exists $E = WOW$ satsifying Hypotheses 1-4 and D such that $F = D \circ_W E$ with $E^t = E$ and $W^t = W$.

Main open problem

Theorem. [McN, van Willigenburg]
Skew diagrams $E_1, E_2, \ldots, E_r$
$E_i = W_i O_i W_i$ satisfies Hypotheses 1-5
E_i' and W_i' denote either E_i and W_i, or E_i^* and W_i^*.
Then

$$((\cdots(E_1 \circ_{W_2} E_2) \circ_{W_3} E_3) \cdots) \circ_{W_r} E_r \sim ((\cdots(E_1' \circ_{W_2'} E_2') \circ_{W_3'} E_3') \cdots) \circ_{W_r} E_r' .$$

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Conjecture. [McN, van Willigenburg; inspired by main result of BTvW]
Two skew diagrams E and E' satisfy $E \sim E'$ if and only if, for some r,

$$\begin{aligned} E &= ((\cdots(E_1 \circ_{W_2} E_2) \circ_{W_3} E_3) \cdots) \circ_{W_r} E_r \\ E' &= ((\cdots(E_1' \circ_{W_2'} E_2') \circ_{W_3'} E_3') \cdots) \circ_{W_r} E_r' , \end{aligned}$$ where

- $\circ$ $E_i = W_i O_i W_i$ satsifies Hypotheses 1-4 for all i,
- $\circ$ E_i' and W_i' denote either E_i and W_i, or E_i^* and W_i^*.

The skew-equivalence class of E will contain 2^r elements, where r is the number of factors E_i in any irreducible factorization of E such that $E_i \neq E_i^*$.

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The skew-equivalence class of E will contain 2^r elements, where r is the number of factors E_i in any irreducible factorization of E such that $E_i \neq E_i^*$.
True for $n \leq 19$.

Summary

- A definition of skew diagram composition. Encompasses the composition, amalgamated composition and staircase operations of RSvW.
- Theorem that generalizes all previous results.
  In particular, explains the 6 missing equivalences from HDL II.
- Conjecture for necessary and sufficient conditions for $E \sim E'$.