From Dyck Paths to Standard Young Tableaux

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Joint work with Juan Gil, Jordan Tirrell, and Michael Weiner

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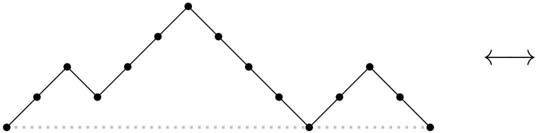
Slides and paper available from
`http://www.unix.bucknell.edu/~pm040/`

Outline



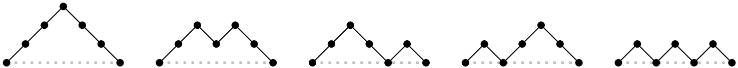
1	2	6
3	4	7
5		

- Background, main question, classic example
- Variations of the classic example
- Hook shapes and flag shapes
- A much more elaborate example

Definitions: Dyck paths

Definition. A Dyck path of semilength n is a sequence of up steps $\mathrm{U} = (1, 1)$ and down steps $\mathrm{D} = (1, -1)$ from $(0, 0)$ to $(2n, 0)$ that stays weakly above the x-axis.

Example. The five Dyck paths of semilength 3.



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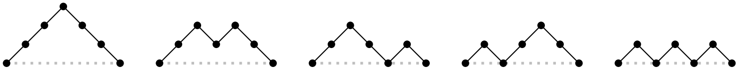


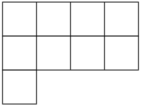
The number of Dyck paths of semilength n is the Catalan number C_n.

Definition. An ascent of a Dyck path is a maximal consecutive sequence of up-steps, and it is a k-ascent if it has length k.

Definitions: standard Young tableaux

Definition. For a partition $\lambda = (\lambda_1, \ldots, \lambda_\ell)$ of n,
a Young diagram of shape λ is an array of boxes left- and top-justified with λ_i boxes in row i.

Example. $\lambda = (4, 4, 1)$



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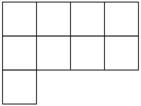

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Definition. A standard Young tableau or SYT is a Young diagram whose n boxes are filled bijectively with $\{1, \ldots, n\}$ such that the entries increase along rows and down columns.

The number of SYT of shape λ is given by the hook-length formula.

The main question

In what ways can we add extra structure or restrictions to Dyck paths and/or SYT to yield equinumerous sets?

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We give 8 ways to answer this question:

0. the classic bijection;

1,2,3. Variations of the classic bijection

4,5,6. Three with the same first step;

7. an elaborate bijection.

Bijection 0. The classic example

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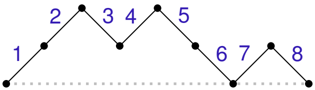
Theorem. Dyck paths of semilength n are in bijection with the SYT of shape (n, n).

Proof. Put indices of U steps in the first row and indices of D steps in the second row.

Example.



$\longleftrightarrow$

1	2	4	7
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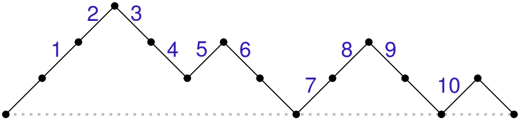
1. Shape $(n, n-d)$

Theorem. [GMTW?] For $0 \leq d \leq n$, Dyck paths of semilength $n+1$ having exactly $d+1$ returns are in bijection with SYT of shape $(n, n-d)$.

Bijection is the same but don't label:

- The first U;
- Any D that touches the x-axis.

Example.



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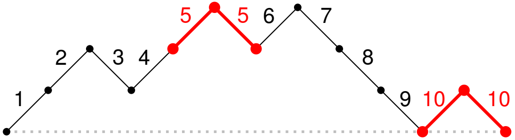
2,3. Marked peaks

Theorem. [GMTW, Similar to result of Pechenik] The number of Dyck paths of semilength n with k marked peaks
equals the number of tableaux of shape (n, n) with label set $\{1, \ldots, 2n-k\}$ such that the rows are strictly increasing and the columns are weakly increasing.

Enumerated by the large Schröder numbers.

Same bijection but use the same label on both steps of a marked peak.

Example.



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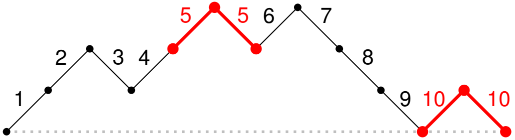
Example. If want strictly increasing columns, need to avoid peaks starting at height 0.



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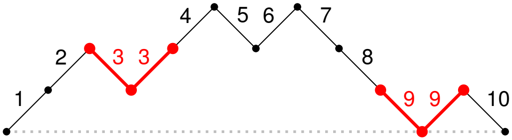
Theorem. [GMTW, Similar to result of Pechenik] The number of Dyck paths of semilength n with k marked peaks (resp. valleys) equals the number of tableaux of shape (n, n) with label set $\{1, \ldots, 2n-k\}$ such that the rows are strictly increasing and the columns are weakly (resp. strictly) increasing.

Enumerated by the large Schröder numbers (resp. small Schröder numbers).

Same bijection but use the same label on both steps of a marked peak/valley.

Example. If want strictly increasing columns, need to avoid peaks starting at height 0. So use valleys instead.



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Bjections using modified tableaux

The next three bijections share crucial first two steps:

Dyck paths $\longleftrightarrow$ nomincreasing set partitions $\longleftrightarrow$ modified tableaux

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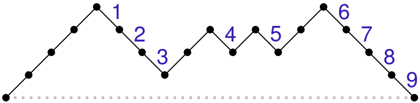
To label U steps:

1. Label the D steps $1, \ldots, n$ from left-to-right.
2. At each peak UD, give the U the same label as the D.
3. Going through the ascents from left-to-right, label the remaining U in a greedy fashion from top-to-bottom.



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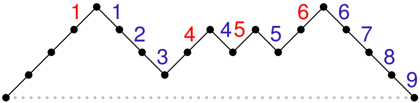

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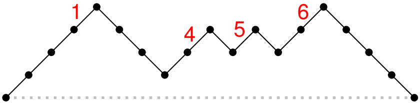

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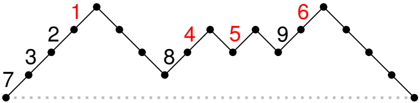

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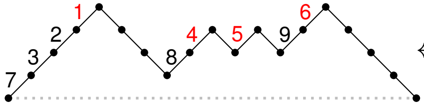

$\longleftrightarrow$ 1237|48|5|69

- Nomincreasing (set) partitions: in standard form, non-minimum entries in each block form an increasing sequence: 23789.

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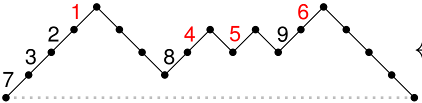

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1	4	5	6
2	8		9
3			
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- Nomincreasing (set) partitions: in standard form, non-minimum entries in each block form an increasing sequence: 23789.
- Modified tableaux: entries increase along first row and down columns; non-first-row entries increase left-to-right.

The main question

Recall the main question:

In what ways can we add extra structure or restrictions to Dyck paths and/or SYT to yield equinumerous sets?

Want bijective proofs that preserve some statistics.

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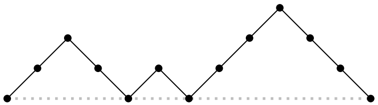
Note. In Bijections 0, 2, 3, #boxes = 2(semilength).
In remaining examples,
#boxes = semilength.

Bijection 1. Hook shapes

Baby Theorem. For $1 \leq k \leq n$, Dyck paths of semilength n with k peaks and k returns are in bijection with SYT of hook shape $(k, 1^{n-k})$.

(1^{n-k} denotes a sequence of $n - k$ copies of 1.)



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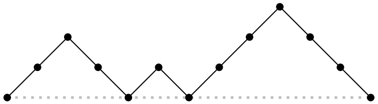

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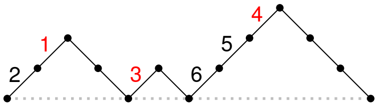

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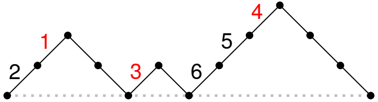

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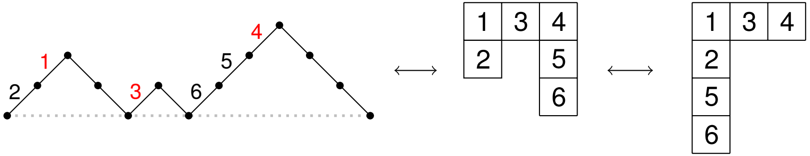

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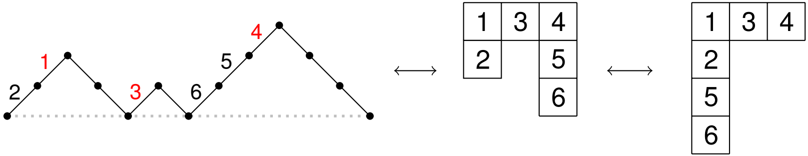

Main idea for inverse direction: In this special situation, the columns of the modified tableau have increasing **consecutive** entries.

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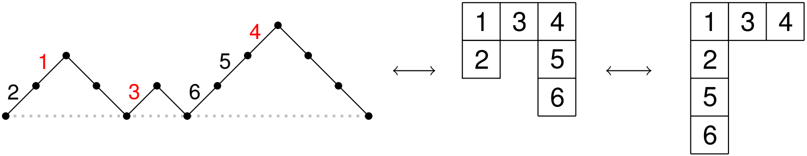

Main idea for inverse direction: In this special situation, the columns of the modified tableau have increasing **consecutive** entries.

Corollary. The number of Dyck paths of semilength n with as many peaks as returns equals the number of SYT of hook shape with n boxes.

Bijection 2: Flag shapes

Definition. An SYT is of flag shape if its shape is $(k, k, 1^{n-2k})$ for some $1 \le k \le \lfloor \frac{n}{2} \rfloor$.

1	3	4	5	9	10	16
2	7	12	13	14	15	17
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Definition. An ascent is a singleton if it has length 1.

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Definition. An ascent is a singleton if it has length 1.

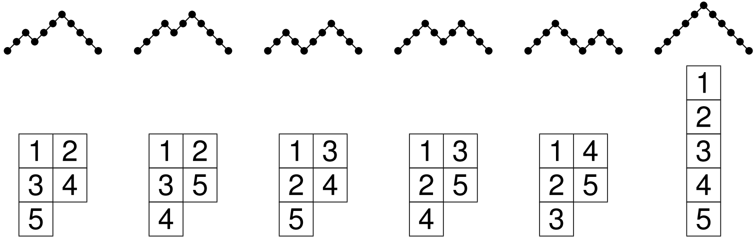
Theorem. The number of Dyck paths of semilength n and no singletons equals the number of SYT of flag shape with n boxes.

These sets are enumerated by the Riordan numbers [A005043].

Bijection 2: Flag shapes

Theorem. The number of Dyck paths of semilength n without singletons equals the number of SYT of flag shape with n boxes.

Example. Let $n = 5$.



Bijection 2: Flag shapes

Theorem. For $1 \leq k \leq \lfloor \frac{n}{2} \rfloor$, Dyck paths of semilength n with k peaks and no singletons are in bijection with SYT of shape $(k, k, 1^{n-2k})$.

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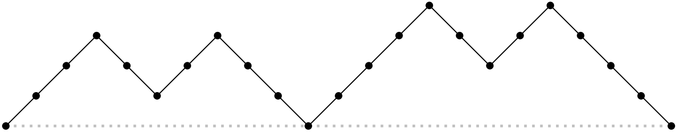
Theorem. For $1 \leq k \leq \lfloor \frac{n}{2} \rfloor$, Dyck paths of semilength n with k peaks and no singletons are in bijection with SYT of shape $(k, k, 1^{n-2k})$.

Proof. By defining modified tableaux, we've done the hard part.



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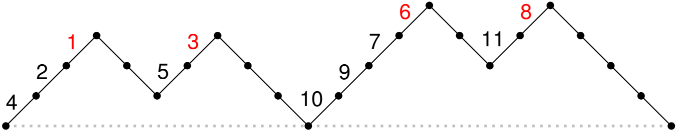

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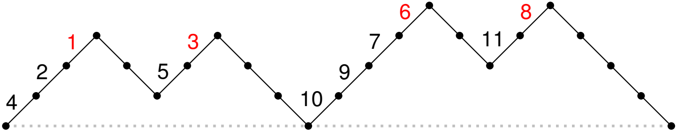

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1	3	6	8
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4		9	
		10	

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For inverse, use: non-first-row entries increase from left-to-right.

Theorem. For $1 \leq k \leq \lfloor \frac{n}{2} \rfloor$, Dyck paths of semilength n with k peaks and no singletons are in bijection with SYT of shape $(k, k, 1^{n-2k})$.

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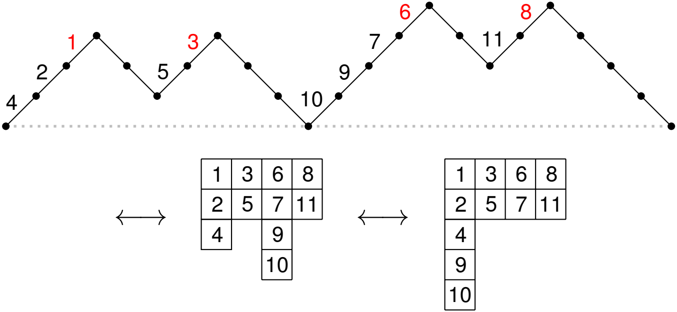

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For inverse, use: non-first-row entries increase from left-to-right.

Corollary. The number of Dyck paths of semilength n without singletons equals the number of SYT of flag shape with n boxes.

Bijection 3: At most 3 rows

Theorem. The number of Dyck paths of semilength n that avoid three consecutive up-steps equals the number of SYT with n boxes and at most 3 rows.

A proof via Motzkin paths already is well known.

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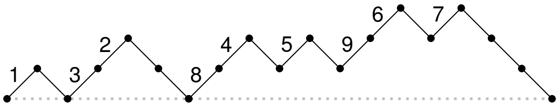

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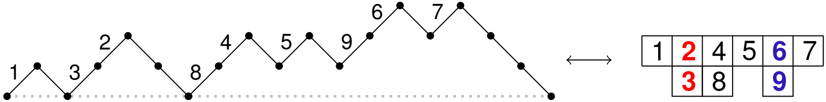

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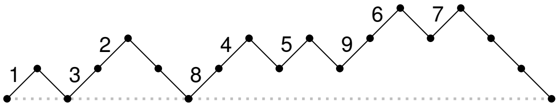

$\longleftrightarrow$

1	**2**	4	5	**6**	7
	3	8		**9**	

$\longleftrightarrow$

1	4	5	7
2	**6**	**9**	
3	**8**		

Bijection 4: All SYT

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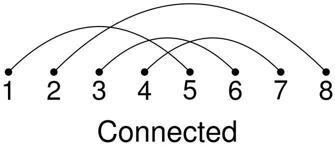
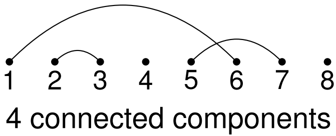
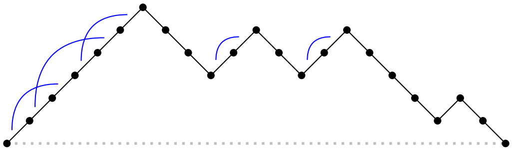
cm-labeled Dyck paths

Definition. A partial matching is connected if the arcs and points form a connected set as a subset of the plane.





Definition. A cm-labeled Dyck path is a Dyck path where each k-ascent is labeled by a connected matching of $[k]$, for every k.

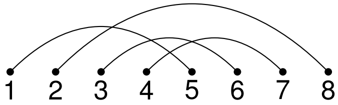


Note. This is both a restriction and additional structure on Dyck paths (ascents lengths must be one or even, but ascents with length at least six have multiple possible labels).

k-noncrossing and k-nonnesting

Theorem. The number of cm-labeled Dyck paths of semilength n with s singletons and k-noncrossing labels equals the number of SYT with n boxes, s odd columns, and at most $2k-1$ rows.

Definition. A k-crossing is a set of k arcs in a partial matching that are pairwise crossing.
We say a partial matching is k-noncrossing if it has no k-crossings.
Similarly for k-nesting and k-nonnesting.



The matching (15)(28)(36)(47) has a 3-crossing (15)(36)(47) but is 4-noncrossing.
It has a 2-nesting (28)(36) or (28)(47) but is 3-nonnesting.

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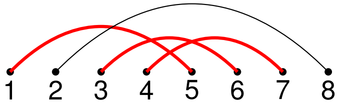

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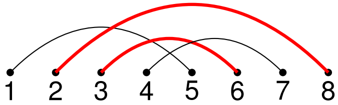

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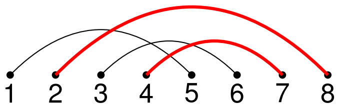

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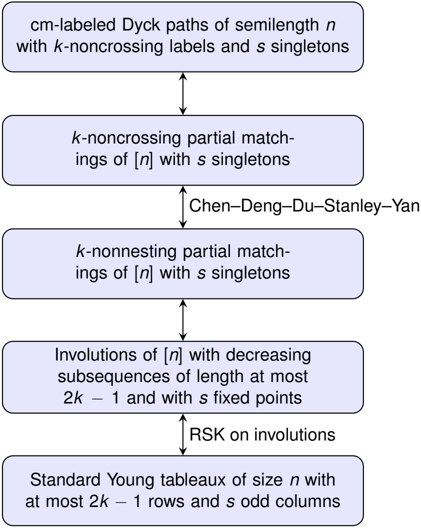
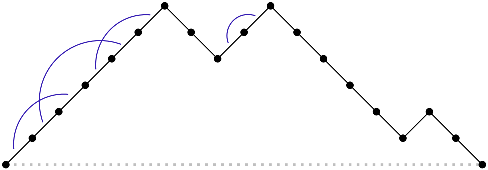


Bijectivity among bottom 4 blocks appears is due independently to Burrill–Courtiel–Fusy–Melczer–Mishna.

cm-labeled Dyck paths to partial matchings

1. Start with a cm-labeled Dyck path



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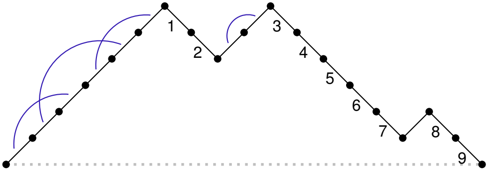

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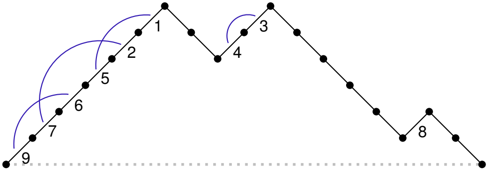

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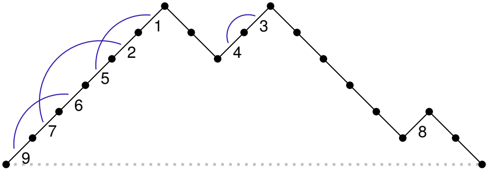

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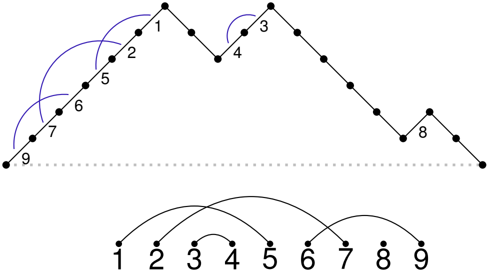

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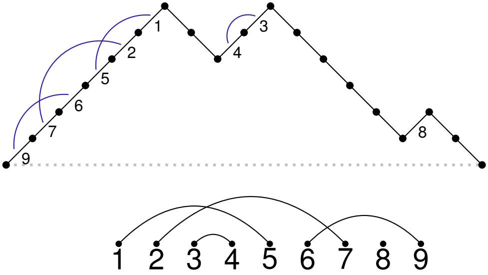

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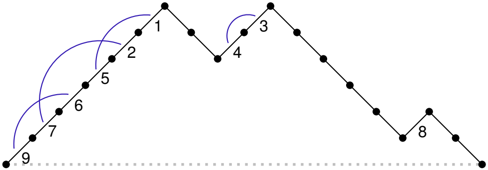
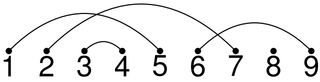
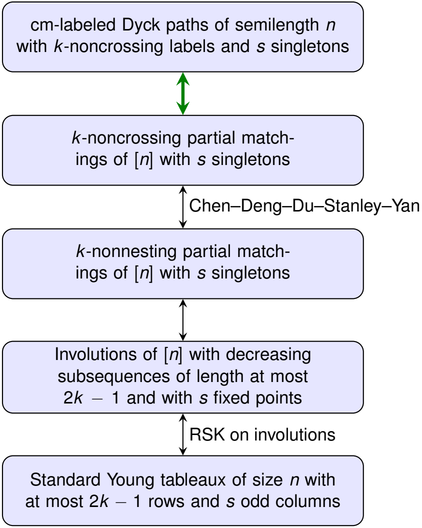




Note. Crossings and singletons preserved.

Inverse: Connected components give ascents. Steps 2–4 give a well-known bijection from unlabeled Dyck paths to non-crossing set partitions.

Structure of the bijection



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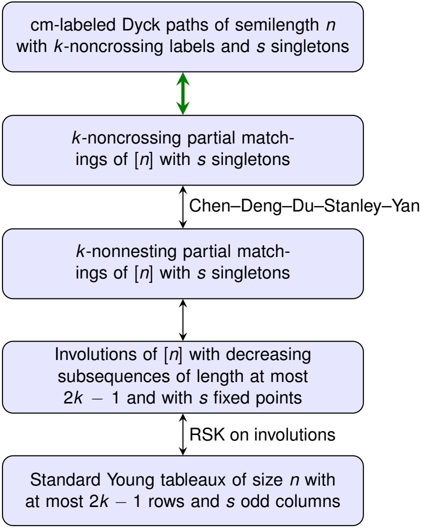
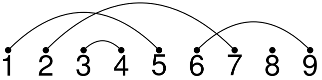


Next: bottom bijection.

Involutions to SYT

First observation. Partial matchings are in bijection with involutions (self-inverse permutations):



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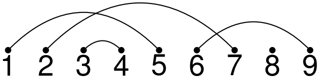

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Robinson–Schensted–Knuth (RSK) Algorithm.

$$\text{permutation} \quad \pi \longleftrightarrow (T, R) \quad \text{two SYT of same shape.}$$

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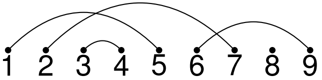

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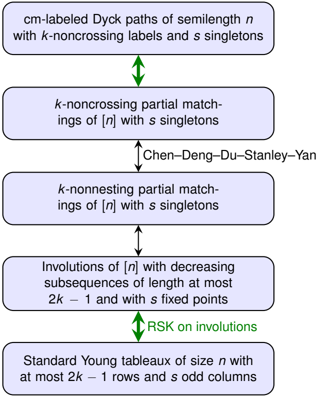
Other facts we need:

- Knuth: # fixed points (singletons) in π = # odd columns in T.
- Schensted:
  Length of longest decreasing subsequence in π = # rows in T.

Structure of the bijection



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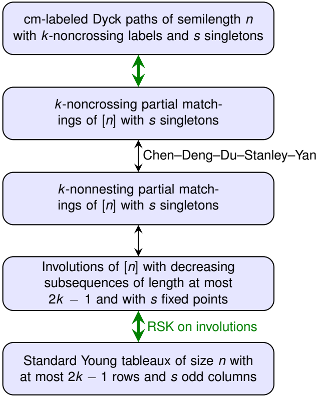

We have:

cm-labeled Dyck paths $\longleftrightarrow$ partial matchings $\longleftrightarrow$ involutions $\longleftrightarrow$ SYT.

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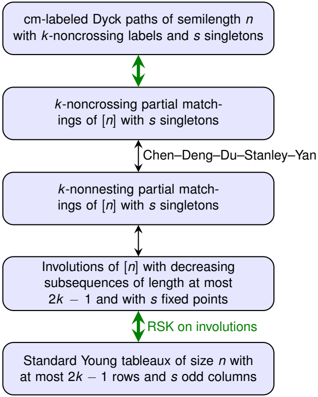

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Nice connection between **nestings** and decreasing subseqences.
Next: k-nesting $\iff$ a decreasing subsequence of length at least $2k$.

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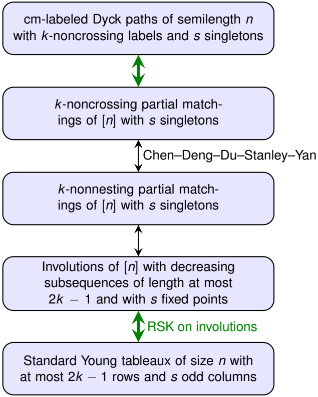

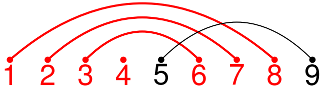

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Structure of the bijection

cm-labeled Dyck paths of semilength n with k-noncrossing labels and s singletons

$\updownarrow$

k-noncrossing partial matchings of $[n]$ with s singletons

$\updownarrow$ Chen–Deng–Du–Stanley–Yan

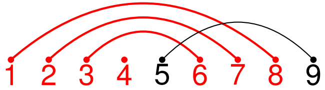
k-nonnesting partial matchings of $[n]$ with s singletons

$\updownarrow$ Post

Involutions of $[n]$ with decreasing subsequences of length at most $2k-1$ and with s fixed points

$\updownarrow$ RSK on involutions

Standard Young tableaux of size n with at most $2k-1$ rows and s odd columns



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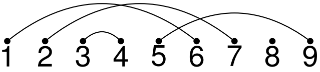
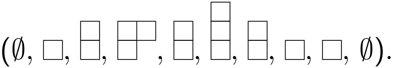
Final step. A bijection from k-noncrossing to k-nonnesting partial matchings of $[n]$ (which preserves singletons).
Chen–Deng–Du–Stanley–Yan: use oscillating tableaux.
We need to use weakly oscillating tableaux.

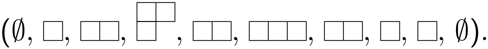
Overview of proof by example. Map the partial matching



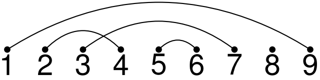
to the weakly oscillating tableau



Take the **transpose**:

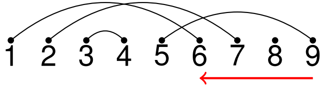


and reverse the map:



The point. k-crossing $\longleftrightarrow$ k-nesting.

Example details.

$M =$ 

j	0	1	2	3	4	5	6	7	8	9
T^j	$\emptyset$	1	1 2	1 3 2	1 2	1 2 5	2 5	5	5	$\emptyset$

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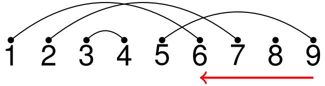
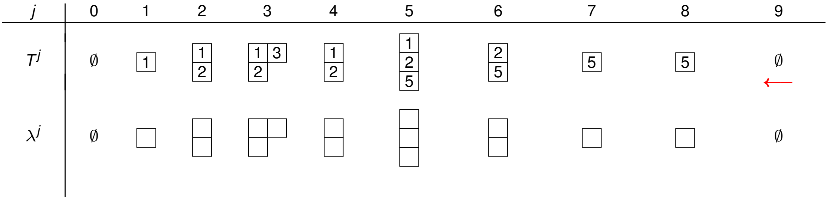
$M =$ 



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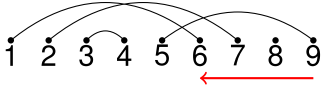
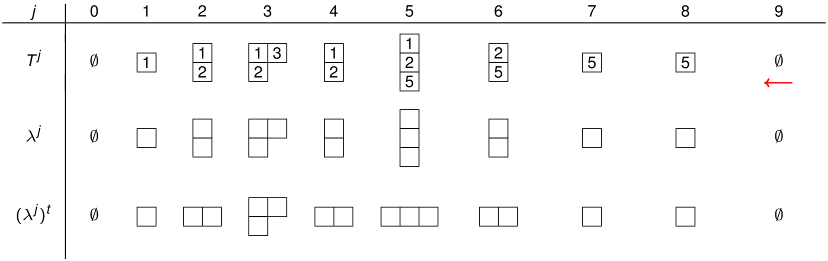
$M =$ 



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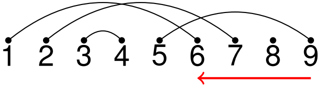
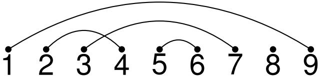
$M =$ 

j	0	1	2	3	4	5	6	7	8	9
T^j	∅	1	1 2	1 3 2	1 2	1 2 5	2 5	5	5	∅
λ^j	∅	□	□ □	□□ □	□ □	□ □ □	□ □	□	□	∅
$(\lambda^j)^t$	∅	□	□□	□□ □	□□	□□□	□□	□	□	∅
$\widehat{T}^j$	∅	1	1 2	1 2 3	1 3	1 3 5	1 3	1	1	∅
$\widehat{M}^j$	∅	∅	∅	∅	(2, 4)	(2, 4)	(2, 4) (5, 6)	(2, 4) (5, 6) (3, 7)	(2, 4) (5, 6) (3, 7)	(2, 4) (5, 6) (3, 7) (1, 9)

$\widehat{M} =$ 

The end

cm-labeled Dyck paths of semilength n with k-noncrossing labels and s singletons

↕

k-noncrossing partial matchings of $[n]$ with s singletons

↕ Chen–Deng–Du–Stanley–Yan

k-nonnesting partial matchings of $[n]$ with s singletons

↕ Post

Involutions of $[n]$ with decreasing subsequences of length at most $2k - 1$ and with s fixed points

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Standard Young tableaux of size n with at most $2k - 1$ rows and s odd columns

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