The Möbius function of generalized subword order

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Joint work with:
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Slides, handout and paper (*Adv. Math.*) available from
`www.facstaff.bucknell.edu/pm040/`

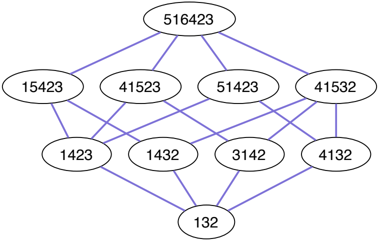
Outline

- Generalized subword order and related posets
- Main result
- Applications
- Mini-tutorial on discrete Morse theory for posets
- How DMT gives the proof of main result

Motivation: Wilf's question

Pattern order: order permutations by pattern containment.

e.g., $4132 \leq 516423$



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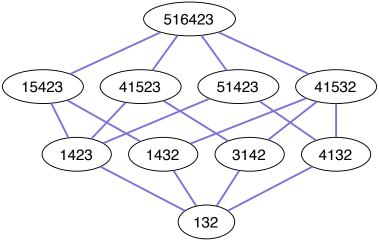

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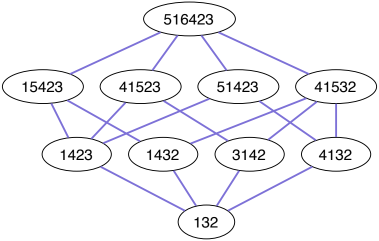

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- Sagan & Vatter (2006)
- Steingrímsson & Tenner (2010)
- Burstein, Jelínek, Jelínková & Steingrímsson (2011)
- Smith (2013)

Still open.

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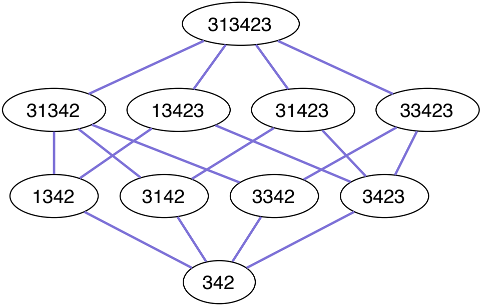
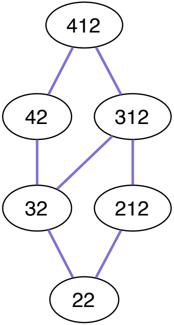
2 partial orders.

1. Subword order.
   A^*: set of finite words over alphabet A.
   $u \leq w$ if u is a subword of w, e.g., 342 $\leq$ 313423.



Motivation for generalized subword order

2. An order on compositions.
$(a_1, a_2, \ldots, a_r) \leq (b_1, b_2, \ldots, b_s)$ if there exists a subsequence $(b_{i_1}, b_{i_2}, \ldots, b_{i_r})$ such that $a_j \leq b_{i_j}$ for $1 \leq j \leq r$.
e.g. $22 \leq 412$.



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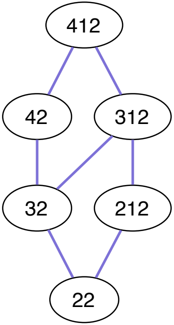

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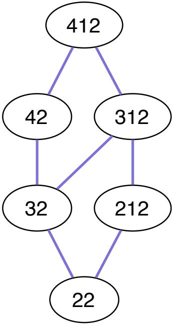

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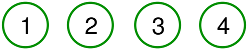
Example 1. If P is an antichain, $u(j) \leq_P w(i_j)$ iff $u(j) = w(i_j)$.



Gives subword order on the alphabet P, e.g., $342 \leq 313423$.

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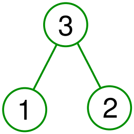
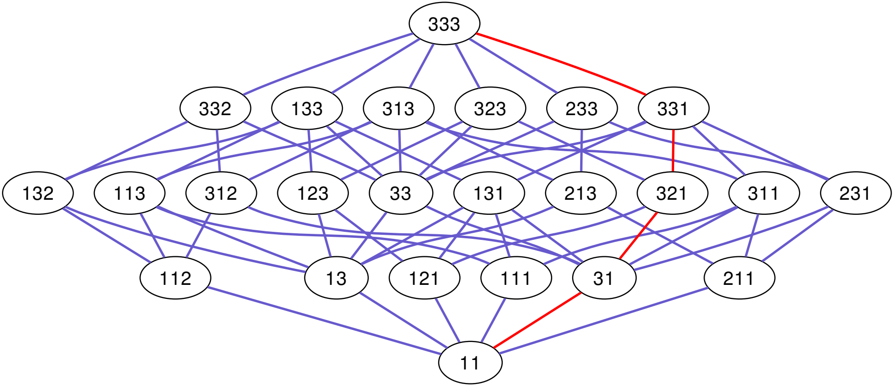
Definition from Sagan & Vatter (2006); appeared earlier in context of well quasi-orderings [Kruskal, 1972 survey].

Example 3. $P = \Lambda$



The interval $[11, 333]$ in P^*:



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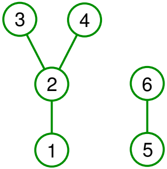

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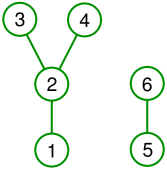

  Includes antichains and chains.
- Sagan & Vatter (2006): when $P = \Lambda$,
  conjecture that $\mu(1^i, 3^j)$ equals certain coefficients of Chebyshev polynomials of the first kind.

  Tomie (2010): proof using ad-hoc methods.

  Our first goal: a more systematic proof.

Main result

P_0: P with a bottom element 0 adjoined.
μ_0: Möbius function of P_0.

Theorem. Let P be a poset so that P_0 is locally finite. Let u and w be elements of P^* with $u \leq w$. Then

$$\mu(u,w) = \sum_{\eta} \prod_{1 \leq j \leq |w|} \begin{cases} \mu_0(\eta(j), w(j)) + 1 & \text{if } \eta(j) = 0 \text{ and} \\ & w(j-1) = w(j), \\ \mu_0(\eta(j), w(j)) & \text{otherwise,} \end{cases}$$

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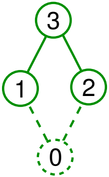

w	$=$	333	333	333
η	$=$	110	101	011

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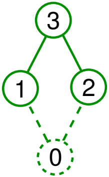

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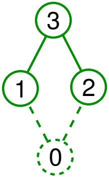

w =	333	333	333	
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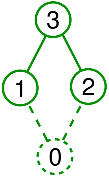

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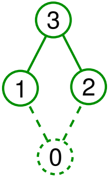

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Application to subword order

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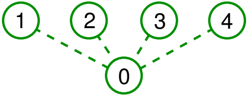

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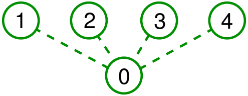

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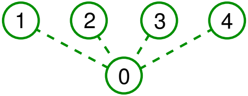

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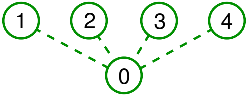

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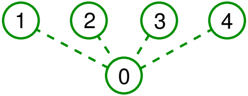

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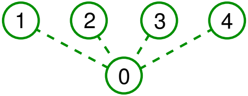

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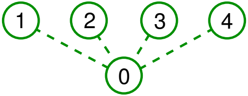

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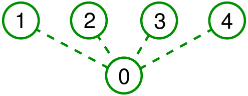

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$$\mu(u, w) = (-1)^{|w|-|u|}(\#\text{ normal embeddings}).$$

More applications

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Application 4. Rederive Tomie's result for $\mu(1^i, 3^j)$ when $P = \Lambda$.

$$\mu(1^i, 3^j) = [x^{j-i}]T_{i+j}(x) \quad \text{for } 0 \le i \le j$$

where $T_n(x)$ is the Chebyshev polynomial of the first kind.

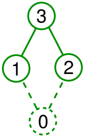
$$T_0(x) = 1, \quad T_1(x) = x, \quad T_n(x) = 2xT_{n-1}(x) - T_{n-2}(x)$$

$$\begin{aligned} T_0(x) &= 1 \\ T_1(x) &= x \\ T_2(x) &= 2x^2 - 1 \\ T_3(x) &= 4x^3 - 3x \\ T_4(x) &= 8x^4 - 8x^2 + 1 \end{aligned}$$

Tomie's result

Restate as

$$\mu(1^i, 3^j) = [x^{j-i}]T_{i+j}(x) = (-1)^i 2^{j-i-1} \frac{i+j}{j} \binom{j}{i}.$$



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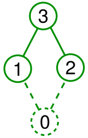

Two types of embeddings of 1^i in 3^j:

33333 10110	Contribution: $(-1)^i 2^{j-i}$	Number: $\binom{j-1}{i-1}$
33333 01101	Contribution: $(-1)^i 2^{j-i-1}$	Number: $\binom{j-1}{i}$

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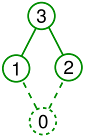


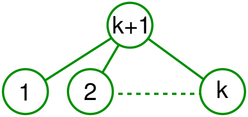
Two types of embeddings of 1^i in 3^j:

33333 10110	Contribution: $(-1)^i 2^{j-i}$	Number: $\binom{j-1}{i-1}$
33333 01101	Contribution: $(-1)^i 2^{j-i-1}$	Number: $\binom{j-1}{i}$

$$\begin{aligned}
\mu(1^i, 3^j) &= (-1)^i 2^{j-i-1}\left(2\binom{j-1}{i-1} + \binom{j-1}{i}\right) \\
&= (-1)^i 2^{j-i-1}\left(\binom{j-1}{i-1} + \binom{j}{i}\right) \\
&= (-1)^i 2^{j-i-1} \frac{i+j}{j}\binom{j}{i}.
\end{aligned}$$

More applications

Application 5. Tomie's results for augmented Λ.



Application 6. Suppose $\mathrm{rk}(P) \leq 1$. Then any interval $[u, w]$ in P^* is

- shellable;
- homotopic to a wedge of $|\mu(u, w)|$ spheres, all of dimension $\mathrm{rk}(w) - \mathrm{rk}(u) - 2$.

Open problem. What if $\mathrm{rk}(P) \geq 2$?

Summary so far

- Generalized subword order interpolates between subword order and an order on compositions.
- For any P, simple formula for the Möbius function of P^* in terms of Möbius function values of P.
- Formula implies all previously proved cases.

Method of proof

- Forman (1995): discrete Morse theory.
- Babson & Hersh (2005): discrete Morse theory for order complexes.
- Sagan & Vatter (2006): concise, accessible exposition.

Take-home message: if the usual methods for determining Möbius functions don't work, try DMT.

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In most of remaining time:
Mini-tutorial on using DMT to determine Möbius functions.

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Take-home message: if the usual methods for determining Möbius functions don't work, try DMT.

In most of remaining time:
Mini-tutorial on using DMT to determine Möbius functions.

We will **not** talk about

- CW-complexes,
- order complexes,
- Morse matchings,

and instead focus on the poset setting.

Step-by-step

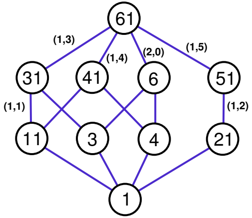
Goal: compute $\mu(x, y)$ using DMT.

1. Pick $\prec$: an ordering of the maximal chains of $[x, y]$ that is a poset lexicographic order (PLO). *Note:* chains are read from top to bottom.
2. Identify the skipped intervals (SIs) of each maximal chain C: an interval I of the interior of C such that $C \setminus I \subseteq B$ for some maximal chain $B \prec C$.
3. Identify the minimal skipped intervals (MSIs) of C: the SIs that are minimal with respect to containment.
4. Remove overlaps among MSIs of C in a certain precise fashion to obtain the set $\mathcal{J}(C)$ of intervals.
5. If the $\mathcal{J}(C)$ cover the interior of C, then C is critical.
6. Compute the Möbius function:

$$\mu(x, y) = \sum_{\text{critical chains } C} (-1)^{|\mathcal{J}(C)|-1}.$$

Example from handout

$P*$:



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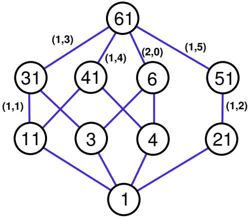

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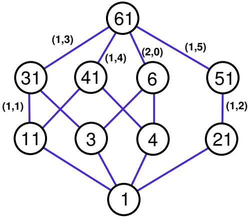

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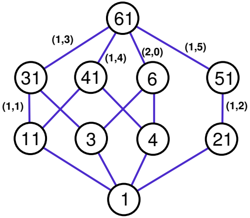

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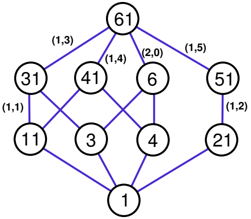

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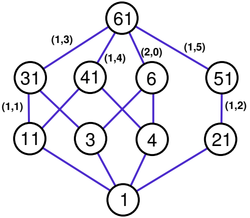

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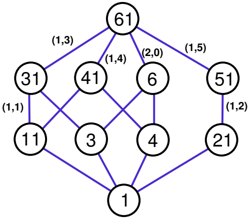

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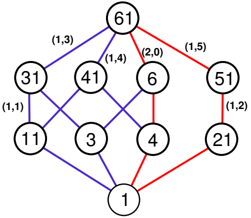

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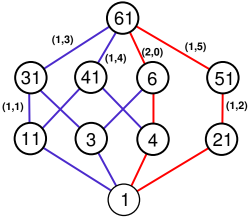

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Poset lexicographic order

Question: When is an ordering $C_1 \prec C_2 \prec \cdots$ of the maximal chains a PLO?

Poset lexicographic order

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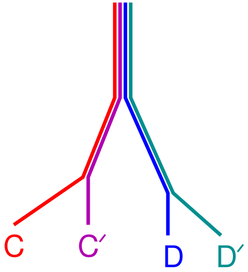
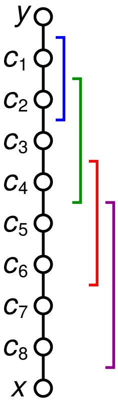
Answer: Whenever the setup in the picture occurs, we require that $C \prec D$ if and only if $C' \prec D'$.



Example. Start with an edge labeling with distinct "down labels" at any element. Then order the maximal chains lexicographically according to its edge labels.

One last technicality...

What if MSIs overlap?



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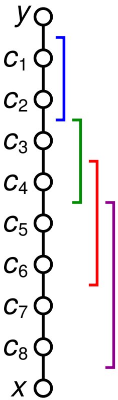

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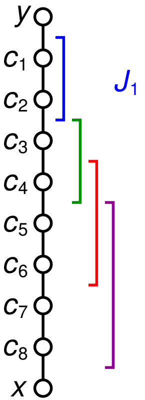

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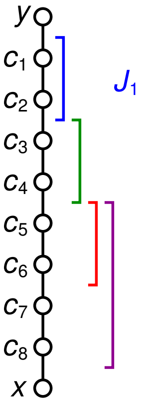

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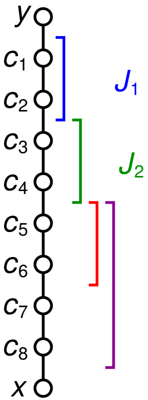

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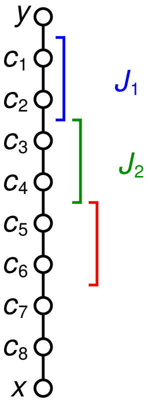

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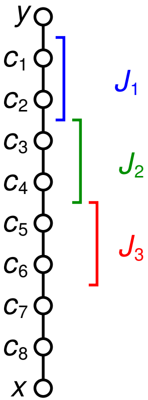

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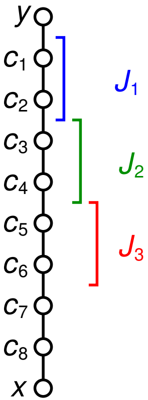

This example is not a critical chain.

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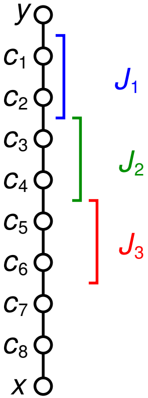

This example is not a critical chain.

For the poset viewpoint of DMT, that's everything!
(great time for questions)

A word or two about the main proof

Theorem. Let P be a poset so that P_0 is locally finite. Let u and w be elements of P^* with $u \leq w$. Then

$$\mu(u, w) = \sum_{\eta} \prod_{1 \leq j \leq |w|} \begin{cases} \mu_0(\eta(j), w(j)) + 1 & \text{if } \eta(j) = 0 \text{ and} \\ & w(j-1) = w(j), \\ \mu_0(\eta(j), w(j)) & \text{otherwise,} \end{cases}$$

where the sum is over all embeddings η of u in w.

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2 key lemmas. Recall: edge labels are (position, element).

1. No MSI can contain an ascent in the position index.
2. A descent in the position index creates a 1-element MSI.

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Upshot. When we reduce w to u along a critical chain:

1. we must work from right to left in w;
2. every time we move left, we create an MSI that won't be involved in any MSI overlaps.

Building critical chains for generalized subword order

$\mu(1, 66)$ illustrates the key ideas.

Case 1: critical chains that end at embedding 01.

$$66 \xrightarrow{(2,\ \ )} \quad \xrightarrow{(2,1)} \boxed{61} \xrightarrow{(1,\ \ )} \qquad \xrightarrow{(1,0)} 01$$

Case 2: critical chains that end at embedding 10.

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lex earlier chain:

$$66 \xrightarrow{(1,3)} 36 \longrightarrow 16 \longrightarrow 06 \longrightarrow 04 \longrightarrow 01$$

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$$66 \xrightarrow{(2,3)} \boxed{63 \xrightarrow{(2,1)} 61} \xrightarrow{(2,0)} \boxed{60} \xrightarrow{(1,4)} \boxed{40} \xrightarrow{(1,1)} 10$$

lex earlier chain:

$$66 \xrightarrow{(1,3)} 36 \longrightarrow 16 \longrightarrow 06 \longrightarrow 04 \longrightarrow 01$$

$\mu(1,66) = 3$.

Building critical chains for generalized subword order

$\mu(1, 66)$ illustrates the key ideas.

Case 1: critical chains that end at embedding 01.

$$66 \xrightarrow{(2,4)} \boxed{64} \xrightarrow{(2,1)} \boxed{61} \xrightarrow{(1,5)} \boxed{51 \xrightarrow{(1,2)} 21} \xrightarrow{(1,0)} 01$$

Case 2: critical chains that end at embedding 10.

$$66 \xrightarrow{(2,5)} \boxed{65 \xrightarrow{(2,2)} 62} \xrightarrow{(2,0)} \boxed{60} \xrightarrow{(1,4)} \boxed{40} \xrightarrow{(1,1)} 10$$

$$66 \xrightarrow{(2,3)} \boxed{63 \xrightarrow{(2,1)} 61} \xrightarrow{(2,0)} \boxed{60} \xrightarrow{(1,4)} \boxed{40} \xrightarrow{(1,1)} 10$$

lex earlier chain:

$$66 \xrightarrow{(1,3)} 36 \longrightarrow 16 \longrightarrow 06 \longrightarrow 04 \longrightarrow 01$$

$\mu(1, 66) = 3$.

13-page proof. Special treatment: $\mu(a0, ab)$ with $a < b$ in P.

Thanks $\longrightarrow \boxed{4} \longrightarrow$ listening!

$[\emptyset, 33333]$ when $P = \lambda$