Cylindric Schur Functions

Retrospective
In
Combinatorics:
Honoring
STANLEY'S 60th
biRth-
Day

24 June 2004

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Slides and forthcoming paper available from
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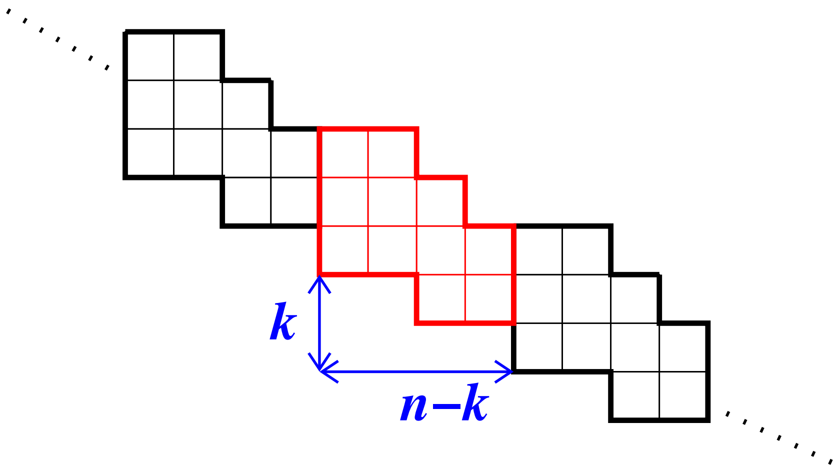
Cylindric skew Schur functions

- Infinite skew shape C
- Invariant under translation
- Identify (x, y) and $(x + k, y - n + k)$.



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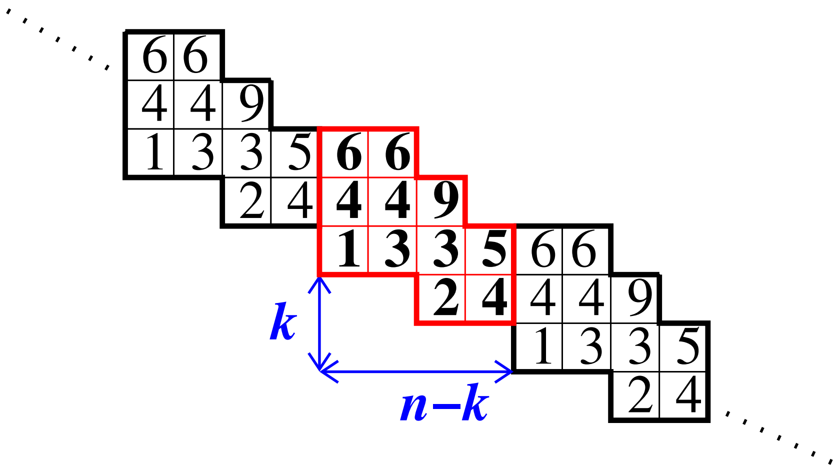


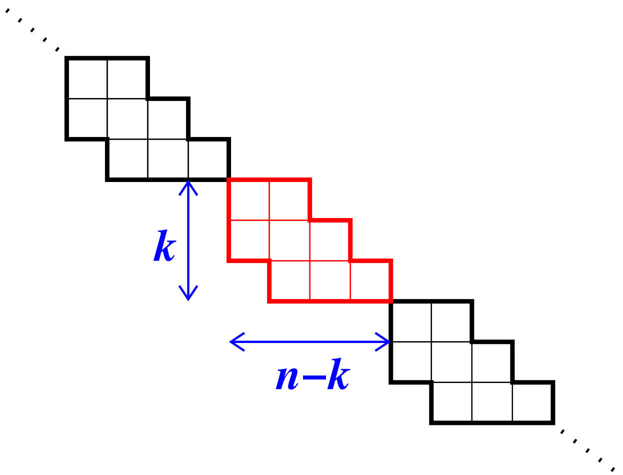
- Entries weakly increasing in each row
  Strictly increasing up each column
- Alternatively: SSYT with relations between entries in first and last columns

$$s_C = \sum_T \mathbf{x}^T = \sum_T x_1^{\#1'\text{s in } T} x_2^{\#2'\text{s in } T} \cdots .$$

Straightforward: s_C is a symmetric function

Cylindric skew Schur functions

EXAMPLE



- Gessel,Krattenthaler: *“Cylindric Partitions”*
- Bertram, Ciocan-Fontanine, Fulton: *“Quantum Multiplication of Schur Polynomials”*
- Postnikov: *“Affine Approach to Quantum Schubert Calculus”* `math.CO/0205165`
- Stanley: *“Recent Developments in Algebraic Combinatorics”* `math.CO/0211114`

Motivation

In $H^*(Gr_{kn})$,

$$\sigma_\lambda \sigma_\mu = \sum_{\nu \subseteq k \times (n-k)} c^\nu_{\lambda\mu} \sigma_\nu.$$

In $QH^*(Gr_{kn})$,

$$\sigma_\lambda * \sigma_\mu = \sum_{d \geq 0} \sum_{\substack{\nu \vdash |\lambda|+|\mu|-dn \\ \nu \subseteq k \times (n-k)}} q^d C^{\nu,d}_{\lambda\mu} \sigma_\nu.$$

$C^{\nu,d}_{\lambda\mu} =$ 3-point Gromov-Witten invariants
$= \#\{$rational curves of degree d in Gr_{kn} that meet fixed generic translates of the Schubert varieties $\Omega_{\nu^\vee}$, Ω_λ and $\Omega_\mu\}$.

Key point: $C^{\nu,d}_{\lambda\mu} \geq 0$.

“Fundamental Open Problem”:

Motivation

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Key point: $C^{\nu,d}_{\lambda\mu} \geq 0$.

"Fundamental Open Problem": Find an algebraic or combinatorial proof of this fact.

What's cylindric got to do with it?

THEOREM *(Postnikov)*

$$s_{\lambda/d/\mu}(x_1,\ldots,x_k) = \sum_{\nu\subseteq k\times(n-k)} C_{\lambda\mu}^{\nu,d} s_\nu(x_1,\ldots,x_k).$$

Conclusion: Want to understand expansions of cylindric skew Schur functions into Schur functions.

COROLLARY $s_{\lambda/d/\mu}(x_1,x_2,\ldots,x_k)$ *is Schur-positive.*

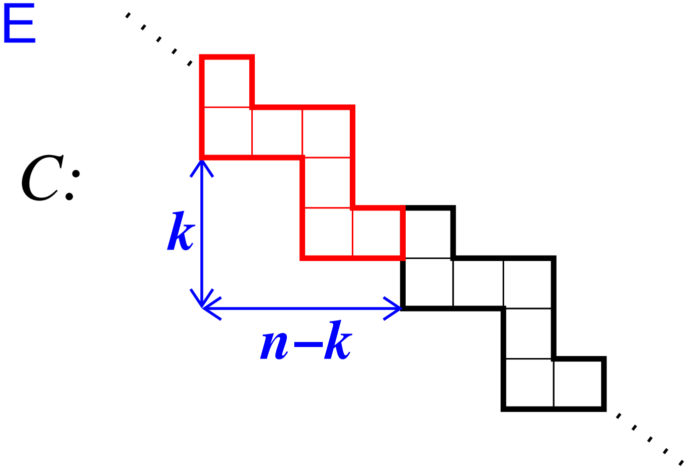
Known: $s_{\lambda/d/\mu}(x_1,x_2,\ldots)$ need not be Schur-positive.

THEOREM *(McN.) For any cylindric shape C,*

$s_C(x_1,x_2,\ldots)$ *is Schur-positive* $\Leftrightarrow$ C *is a skew shape*.

Example: Cylindric ribbons

EXAMPLE

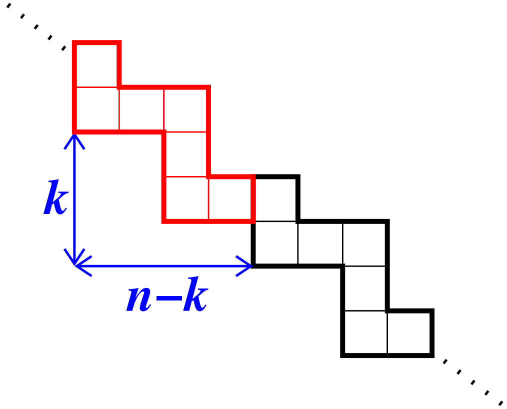


$$s_C(x_1, x_2, \ldots) \;=\; \sum_{\nu \subseteq k \times (n-k)} c_\nu s_\nu \;+ s_{n-k,1^k} - s_{n-k-1,1^{k+1}}$$
$$+ s_{n-k-2,1^{k+2}} - \cdots + (-1)^{n-k} s_{1^n}.$$

Schur-positive with $k+1$ variables
Not Schur-positive with $\geq k+2$ variables
General cylindric skew shape: $\geq k+2+l$ variables
Toric shapes: $\geq 2k+1$ variables

Example: Cylindric ribbons

C:



$$s_C(x_1, x_2, \ldots) \quad = \quad \sum_{\nu \subseteq k \times (n-k)} c_\nu s_\nu \;\; + s_{n-k,1^k} - s_{n-k-1,1^{k+1}}$$

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Example: Cylindric ribbons

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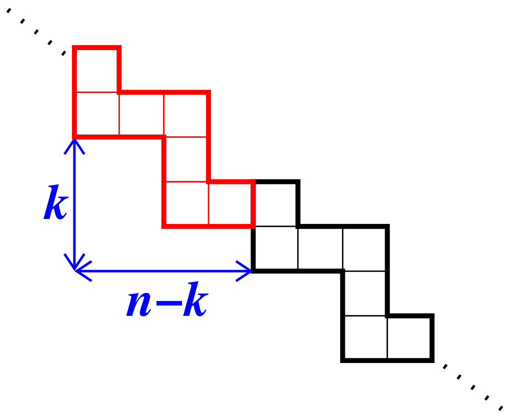
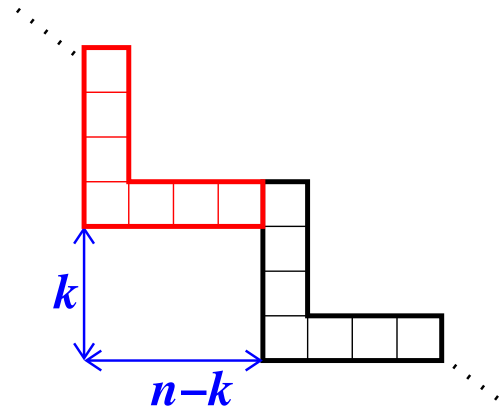


H:



$$s_C(x_1, x_2, \ldots) \;=\; \sum_{\nu \subseteq k \times (n-k)} c_\nu s_\nu \; + s_{n-k,1^k} - s_{n-k-1,1^{k+1}} + s_{n-k-2,1^{k+2}} - \cdots + (-1)^{n-k} s_{1^n}.$$

However, $s_C(x_1, x_2, \ldots) = \displaystyle\sum_{\nu \subseteq k \times (n-k)} c_\nu s_\nu \; + s_H.$

s_C: cylindric *skew* Schur function

s_H: cylindric Schur function

We say that s_C is cylindric Schur positive.

A Conjecture

CONJECTURE *For any cylindric shape C, s_C is cylindric Schur positive.*

Tool: Cylindric skew Schur functions as alternating sums of skew Schurs

Bertram, Ciocan-Fontanine, Fulton:

☺ Nice description in terms of ribbons

☹ Only for toric shapes, certain terms

Gessel, Krattenthaler:

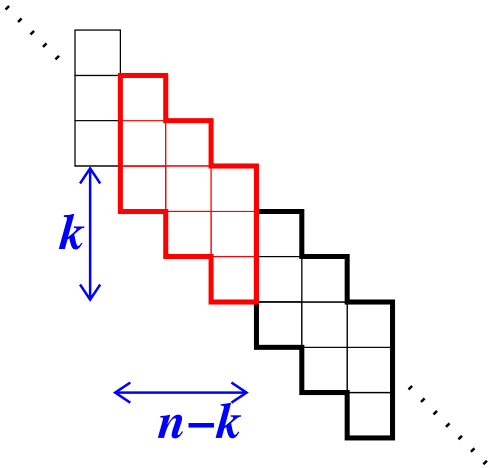
☺ Works for all cylindric shapes

☹ Not as nice a description

We can get the best of both worlds:
A technique for expanding a cylindric skew Schur function in terms of skew Schur functions that
Works for all cylindric shapes like G-K and
has a nice description like B-CF-F

Tool: Cylindric skew Schur functions as alternating sums of skew Schurs

Example



Tool: Cylindric skew Schur functions as alternating sums of skew Schurs

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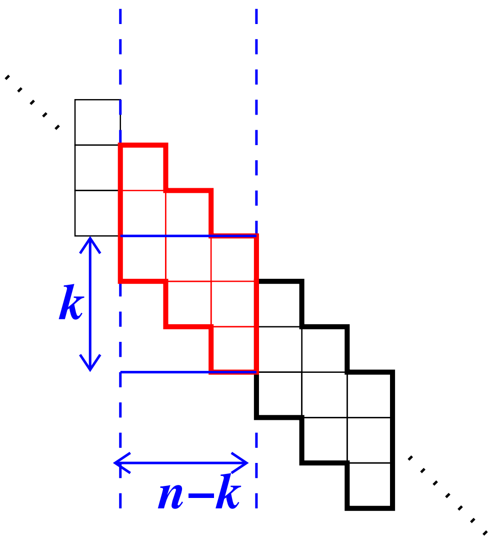

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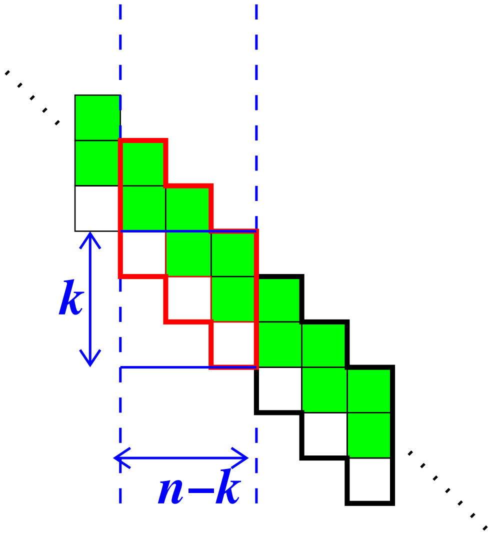

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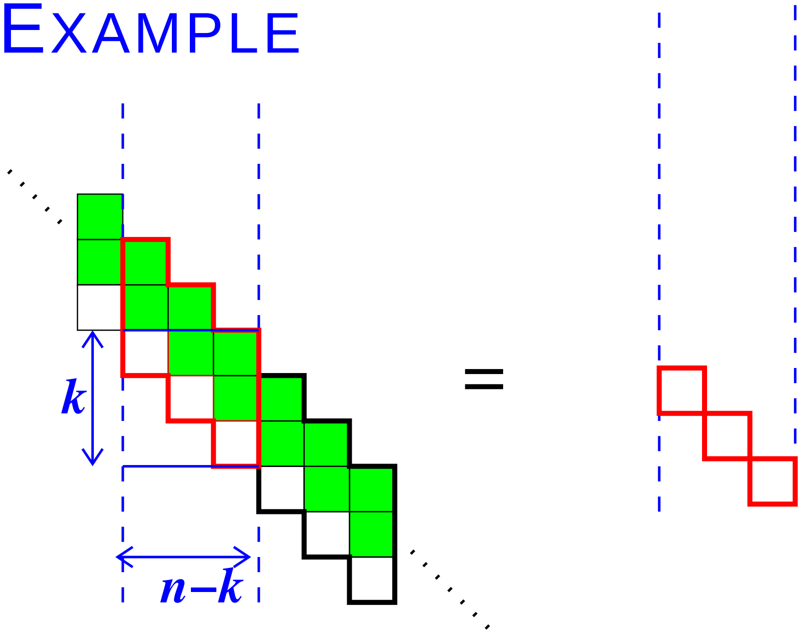

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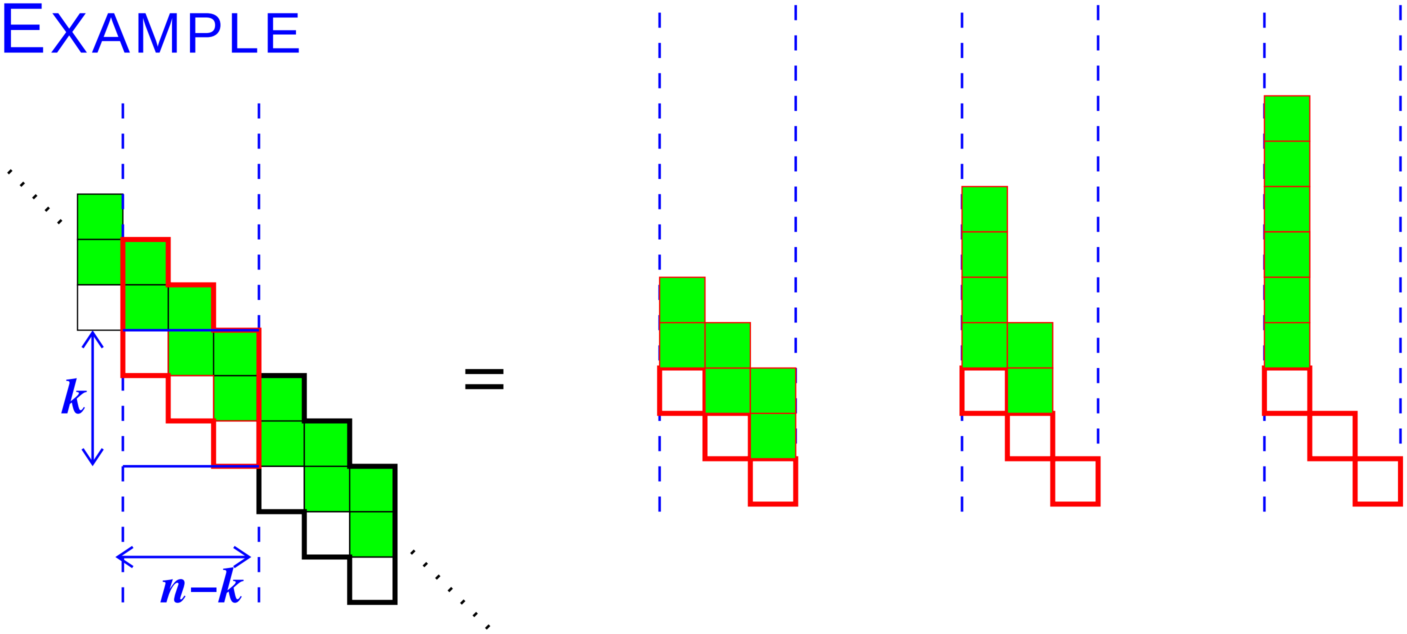
Example



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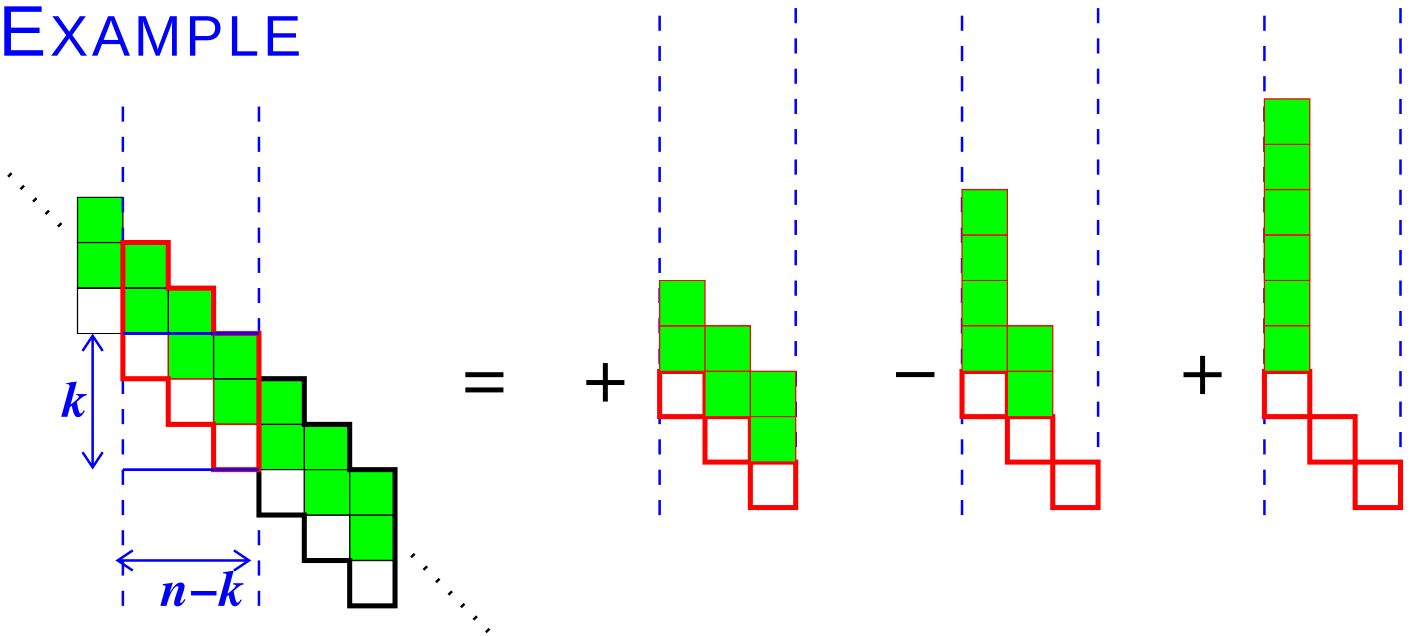

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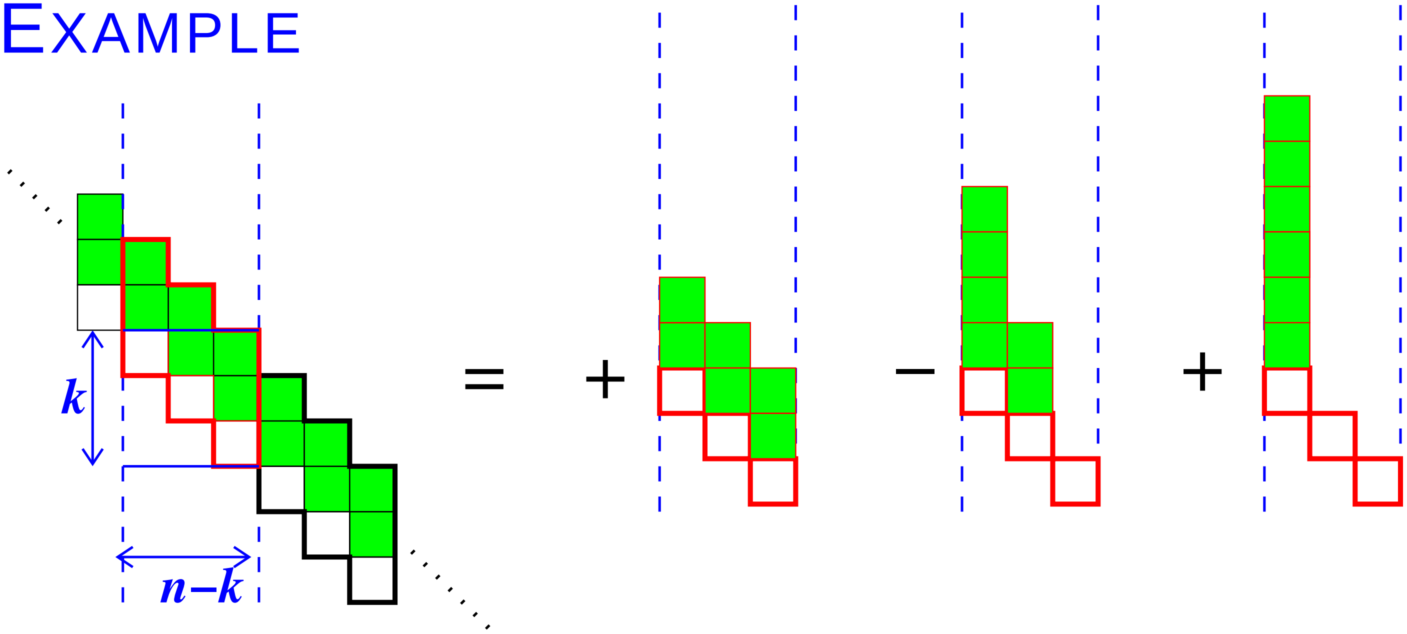
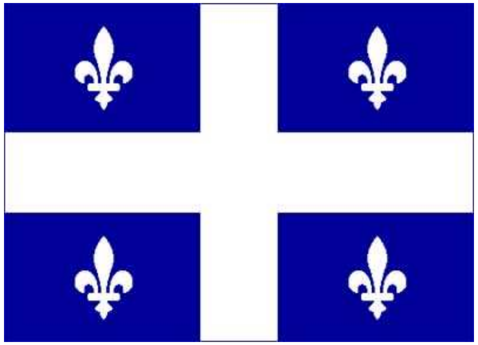


$$s_C = s_{33321/21} - s_{3222111/21} + s_{321111111/21}$$

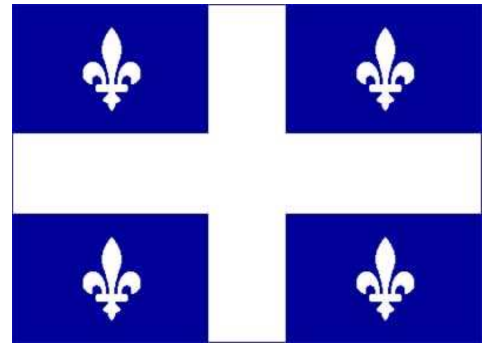
$$= s_{333} + 2s_{3321} + s_{33111} + s_{3222} - s_{321111} + s_{3111111}$$

$$-s_{22221} - 2s_{222111} + 2s_{21111111} + s_{111111111}.$$

St.-Jean-Baptiste Day



St.-Jean-Baptiste Day



Special Session in Algebraic Combinatorics
Canadian Mathematical Society Winter Meeting
Saturday, December 11 - Monday, December 13
McGill University, Montréal

`http://www.lacim.uqam.ca/~biagioli/CMS/cms.html`

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