An Introduction to Partially Ordered Sets

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Slides available from

`http://www-math.mit.edu/~mcnamara/`

Definition A *partially ordered set* or *poset* P is a set S, together with a relation $\leq$, with the following properties:

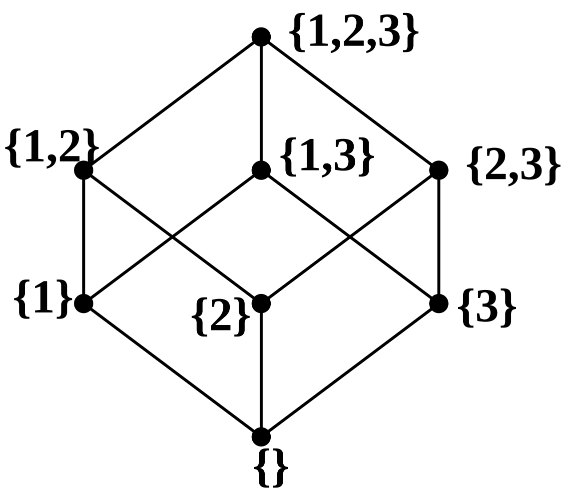
1. *Reflexivity:* $x \leq x$ for all $x \in P$
2. *Antisymmetry:* If $x \leq y$ and $y \leq x$ then $x = y$
3. *Transitivity:* If $x \leq y$ and $y \leq z$ then $x \leq z$

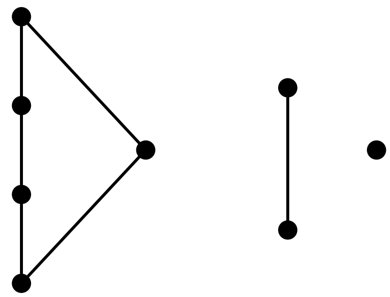
EXAMPLE The set of subsets of $\{1, 2, 3\}$ ordered by containment. Then, for example, $\emptyset \leq \{1, 3\} \leq \{1, 2, 3\}$.

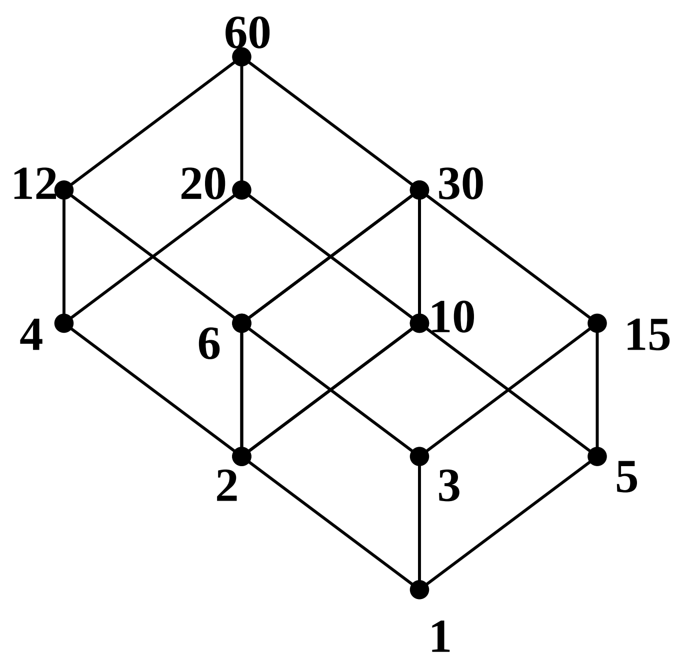
EXAMPLE The set of divisors of 60 where $\leq$ corresponds to "divides". Then, for example, $6 \leq 30$ but $6 \not\leq 15$.

If $x < y$ in P, then we say that y *covers* x if there's no z with $x < z < y$.

Then we can draw the *Hasse diagram* of P where we draw an edge from x up to y if and only if y covers x.



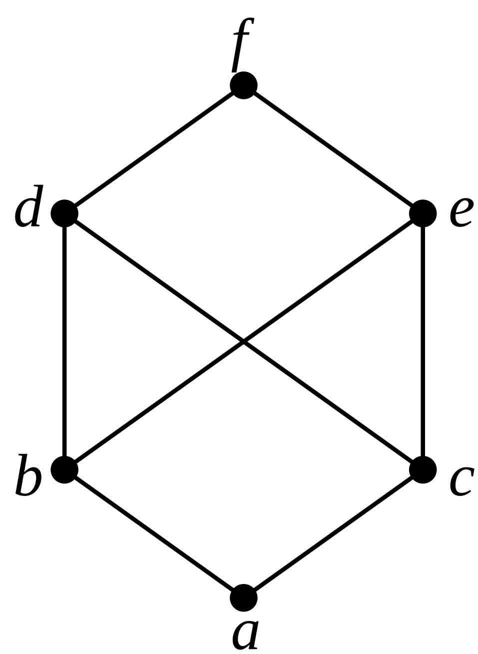




Definition A poset P is said to be a *lattice* if every two elements x and y of P have a least upper bound and a greatest lower bound.

We call the least upper bound the *join* of x and y and denote it by $x \vee y$.

We call the greatest lower bound the *meet* of x and y and denote it by $x \wedge y$.



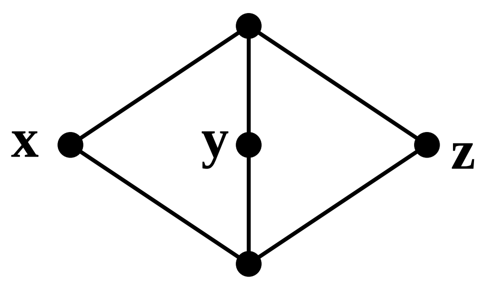
We say that a lattice L is *distributive* if

$$x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$$

and

$$x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$$

for all elements x, y and z of L.



Consider EXAMPLE The lattice of *order ideals* of a poset P.

The Fundamental Theorem of Finite Distributive Lattices, due to Birkhoff:

THEOREM *A lattice L is distributive if and only if it is the lattice of order ideals of some poset P.*

We write $L = J(P)$.

An edge-labeling of a poset P is said to be an *EL-labeling* if it satisfies the following 2 conditions:

1. Every interval $[x, y]$ of P has exactly one maximal chain with increasing labels
2. This chain has the lexicographically least set of labels

Who cares?

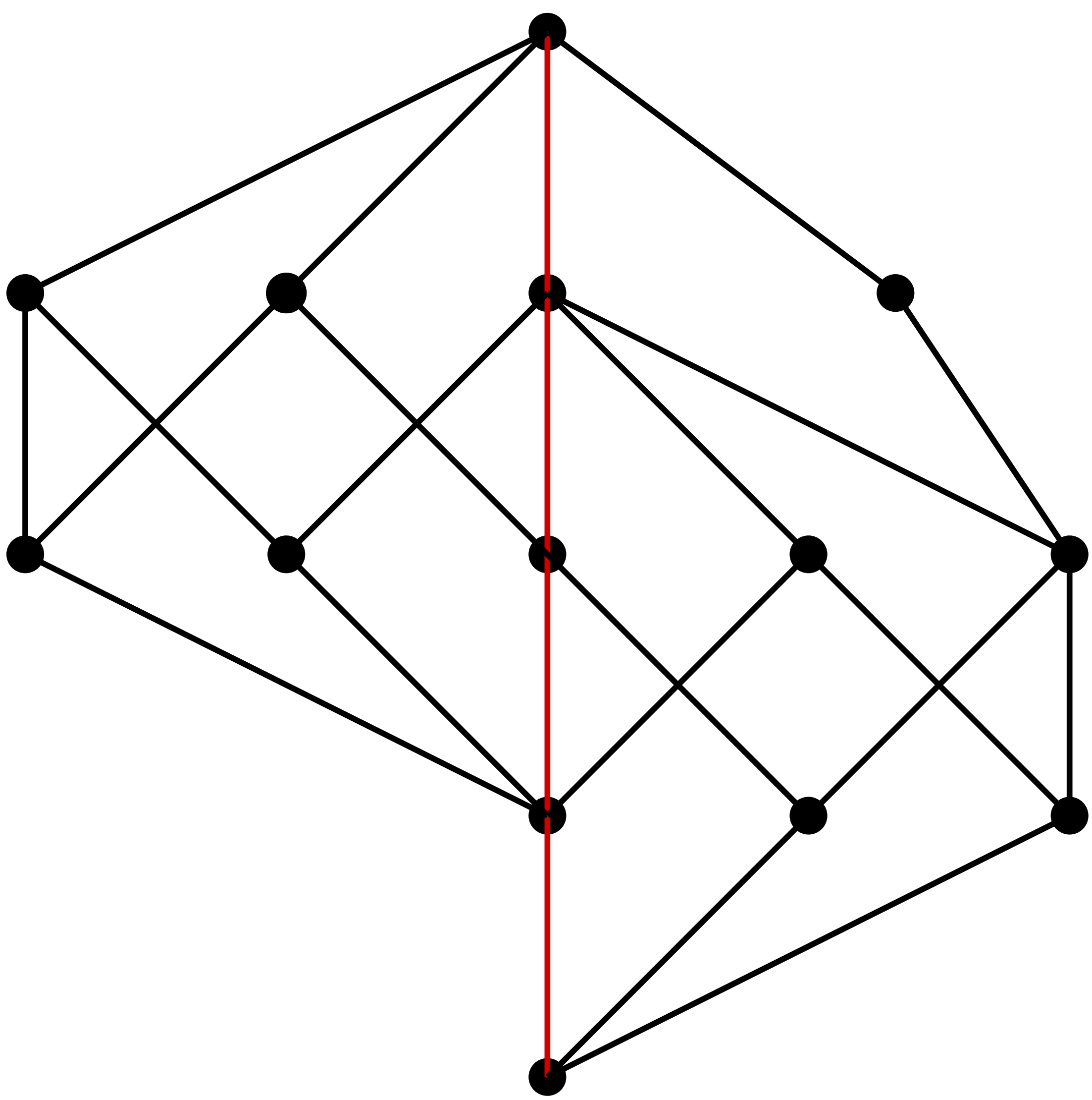
- EL-labeling $\Rightarrow$ Shellable $\Rightarrow$ Cohen-Macaulay
- 1980 MathSciNet hits for Cohen-Macaulay

How far can we take our method for giving a poset an EL-labeling?

Are there other classes of posets that have an EL-labeling?

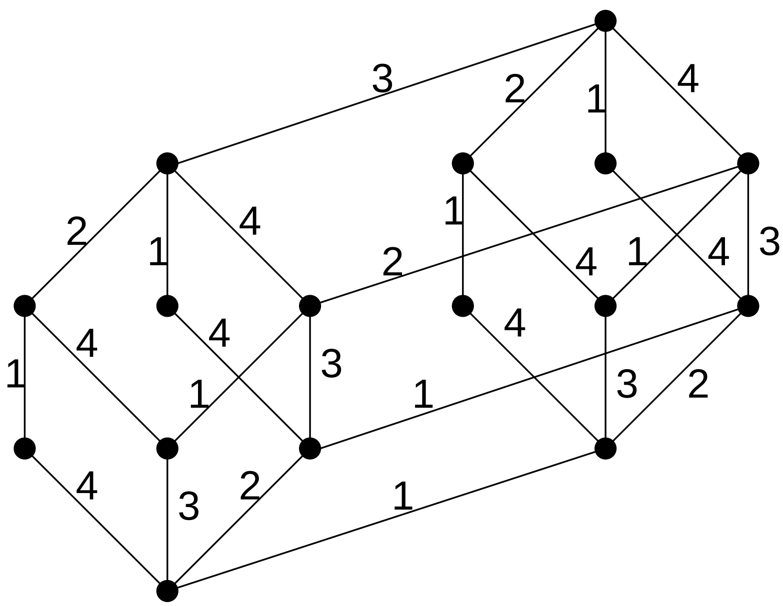
Definition A finite lattice L is said to be *supersolvable* if it contains a maximal chain $\mathfrak{m}$, called an *M-chain* of L which together with any other chain of L generates a distributive sublattice.

EXAMPLE



Remark Our EL-labeling of supersolvable lattices have the additional nice property that the labels along any maximal chain give a permutation.

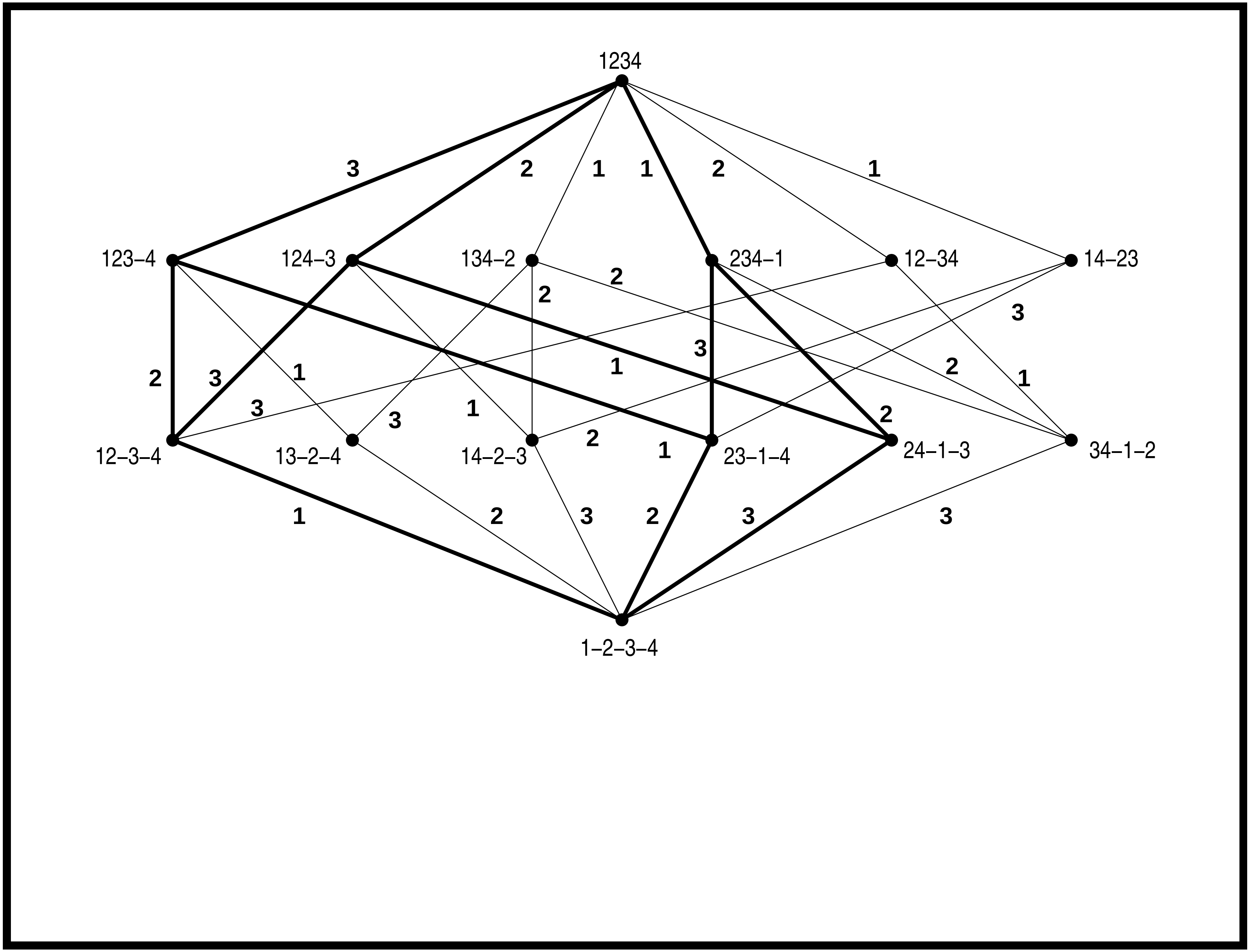
In this case, we call our labeling an *S_n EL-labeling* or *snelling*, for short.



THEOREM *A lattice is supersolvable if and only if it has an S_n EL-labeling.*

We want the chain $\mathfrak{m}_0$ with labels $1, 2, 3, \ldots, n$ to be an M-chain. Let $\mathfrak{m}$ be any other chain of L. (It suffices to consider only maximal chains.) The proof relies on the equivalence of the following 3 posets:

1. The sublattice of L generated by $\mathfrak{m}$ and $\mathfrak{m}_0$
2. Let $\omega_\mathfrak{m}$ be the permutation labeling $\mathfrak{m}$. We construct a poset $P_{\omega_\mathfrak{m}}$ on the numbers $1, 2, \ldots, n$ by drawing an edge from i up to j if $i < j$ and i and j are the "right way around" in $\omega_\mathfrak{m}$. Then we construct and label $J(P_{\omega_\mathfrak{m}})$ as before.
3. If $\mathfrak{m}$ has a *descent* at i, then we define $S_i(\mathfrak{m})$ to be the unique chain in L differing from $\mathfrak{m}$ only at level i and having no descent at i. If $\mathfrak{m}$ doesn't have a descent at i then we set $S_i(\mathfrak{m}) = \mathfrak{m}$. We define $Q_\mathfrak{m}$ to be the "closure" of $\mathfrak{m}$ in L under the action of $S_1, S_2, \ldots, S_{n-1}$.


1234
123–4
124–3
134–2
234–1
12–34
14–23
12–3–4
13–2–4
14–2–3
23–1–4
24–1–3
34–1–2
1–2–3–4

The action of $S_1, S_2, \ldots, S_{n-1}$ has the following properties:

1. It is a local action: it only changes a chain in one place
2. ${S_i}^2 = S_i$
3. $S_i S_j = S_j S_i$ if $|i - j| \geq 2$
4. $S_i S_{i+1} S_i = S_{i+1} S_i S_{i+1}$
5. $\mathrm{ch}(\chi(x)) = \omega(F_L(x))$
   In other words, choose any subset S of $1, 2, \ldots, n-1$. Then the number of chains of L stopping only at the levels in S equals the number of maximal chains fixed under S_i for all $i \notin S$.

An action on the maximal chains of a lattice having all of these properties is called a *good $\mathcal{H}_n(0)$ action*.

Theorem *Let L be a finite lattice. TFAE:*

1. *L is supersolvable*
2. *L has an S_n EL-labeling*
3. *L has a good $\mathcal{H}_n(0)$ action*